9 Lectures + 2 Examples Classes **KMK**

Plasticity and plastic flow in crystals (2 lectures)

Plastic flow and its microscopic and macroscopic descriptions. Yield in crystals: derivation of the stress tensor for glide on a general plane in a general direction. Definition of an independent slip system. Need for five independent slip systems to accommodate general strain. Examples in metals and ceramics.

Continuum plasticity (2 lectures)

Stress-strain curves of real materials. Definition of yield criterion. Concept of a yield surface in principal stress space. Yield criteria: Tresca, von Mises, Coulomb, pressure-modified von Mises. Physical interpretation of Tresca (maximum shear stress) and von Mises (strain energy density, octahedral shear stress) yield criteria. Experimental test of yield criteria (Taylor and Quinney). Yield criteria applicable to metals, polymers and geological materials.

Plastic strain analyses (2 lectures)

Lévy-Mises equations. Example: pressurised thin-walled cylinder. Deformation in plane stress: yielding of thin sheet in biaxial and uniaxial tension. Plane stress deformation: Lüders bands. Plane strain deformation: derivation of stress tensor and separation into hydrostatic and deviatoric components. Equivalence of Tresca and von Mises yield criteria in plane strain.

Slip line field theory (1 lecture)

Physical interpretation of slip lines. Slip line fields for compression of slab, comparison with shear pattern in transparent polymers. Hencky relations. Examples: indentation by flat punch, yield in deeply notched bar.

Estimation of forces for plastic deformation (2 lectures)

Stress evaluation and work formulae. Application to rolling and wire-drawing. Upper and lower-bound theorems. Limit analyses for indentation, extrusion, machining and forging. Velocity-vector diagrams: hodographs. Classical theory of the strength of soldered and glued joints. Analysis of forging with friction. Rolling loads and torques: analogy with forging. Use of finite-element methods in analyses of metalforming operations.

Book List

P.P. Benham, R.J. Crawford and C.G. Armstrong, *Mechanics of Engineering Materials*, 2nd edn, Prentice-Hall, 1996	AB181
C.R. Calladine, *Plasticity for Engineers*, Ellis Horwood, 1985	Kc38
G.E. Dieter, *Mechanical Metallurgy*, McGraw-Hill, 1988	Ka62
W.F. Hosford and R.M. Caddell, *Metal Forming*, 3rd. edn., Cambridge Univ. Press, 2007	Ga177b
W. Johnson and P.B. Mellor, *Engineering Plasticity*, Van Nostrand Reinhold, 1983	Kc29
A. Kelly, G.W. Groves and P. Kidd, *Crystallography and Crystal Defects*, John Wiley and Sons, 2000	NbA84

Other books in the Departmental library that you might find useful are:

W.A. Backofen, *Deformation Processing*, Addison-Wesley, 1972	Ga96
R. Hill, *The Mathematical Theory of Plasticity*, Clarendon Press, 1950	Kc3
G.W. Rowe, *Principles of Industrial Metalworking Processes*, Edward Arnold, 1977	Ga106
G.W. Rowe, *Elements of Metalworking Theory*, Edward Arnold, 1979	Ga124
G.W. Rowe, C.E.N. Sturgess, P. Hartley and I. Pillinger, *Finite-element Plasticity and Metalforming Analysis*, CUP, 1991	Kc42

In addition there are three teaching and learning packages relevant to this course in the Plasticity and Deformation Processing part of the Teaching and Learning Packages Library at http://www.doitpoms.ac.uk:

http://www.msm.cam.ac.uk/doitpoms/tlplib/metal-forming-1/index.php
http://www.msm.cam.ac.uk/doitpoms/tlplib/metal-forming-2/index.php
http://www.msm.cam.ac.uk/doitpoms/tlplib/metal-forming-3/index.php

covering the topics of

Stress Analysis and Mohr's Circle
Introduction to Deformation Processes
Analysis of Deformation Processes

respectively. These are based on the C12 lecture course and should help to reinforce ideas that you will meet in this course about stress, strain, yield, yield criteria, metal forming processes and energy estimates for deformation processes.

C12: Plasticity and Deformation Processing

This is a nine lecture course with two Examples Classes.

The course follows on very closely from C4 (Tensor Properties), in which stress and strain tensors were considered in some depth and applied to **elastic** deformation. For example, we will be using the Mohr's circle construction in this course. We shall also use some concepts first introduced in Part IB in the mechanical properties course.

The content of the course can be described as a description and analysis of the **plastic** flow of materials, together with an analysis of plastic deformation in the context of materials deformation. The subject content of the course is not specific to particular materials – it is generic.

Of the various books recommended in the reading list, the book by Dieter is a useful general book and the book by Kelly, Groves and Kidd is very useful for the material in the first two lectures. The books by Calladine, Hosford and Caddell, and Johnson and Mellor are source texts for the deformation processing part of the course.

Plasticity and Plastic Flow

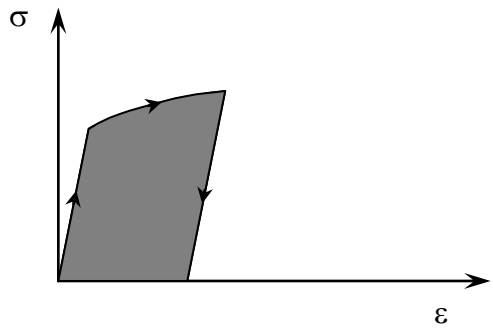
In C4 **elastic** stresses and **elastic** strains, i.e., recoverable σ and ε, were considered. Viscoelastic materials such as polymers can have time-dependent σ and ε, while still remaining in the elastic region of deformation.

Thus, for a Hookean solid, we have

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{in general}$$

$$\sigma = E\varepsilon \quad \text{for an isotropic solid in uniaxial tension}$$

If we increase the strain ε, beyond the elastic limit, we find that it is **not** wholly recoverable. For metals we get a stress-strain curve like:



The shaded area is the work done per unit volume, i.e, the energy dissipated per unit volume.

We can have large strains involved, e.g., tens of % in a metal and we have **irrecoverable deformation**. Hence the process is **irreversible**. This defines **plastic flow**.

Most of the work (> 95%) is dissipated as heat – your everyday experience will remind you that metals heat up if they are plastically deformed.

There are two main approaches to analyse plastic deformation in materials – the **microscopic** approach and the **macroscopic** approach, both of which you have already met before at Part IA and Part IB.

Microscopic approach

You first met this in Part IA in connection with dislocation glide in metal single crystals and polycrystals.

Initially, this approach considers the mechanism of plastic flow in an individual crystal, e.g., in a metal, this relates to the slipping of the material on glide planes by the passage of dislocations. We can learn a lot about the behaviour of a material from this approach, but we cannot use it to design a pressure vessel or a forging press! It is also very difficult to extend an understanding of **single crystal** plastic deformation to that of a **polycrystalline** metal.

Hence, in parallel with the microscopic approach, we have the

Macroscopic approach

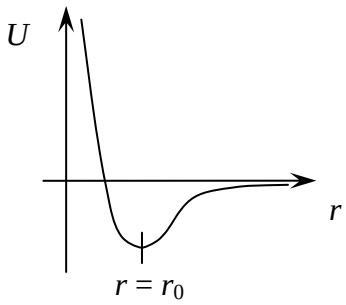
Here the material is treated, in the first approximation, as an isotropic continuum and its constitutive equations are defined as for the case of elasticity.

With this approach, we do not have to worry about slip directions and slip planes. We treat metals as perfect materials characterised by only a few parameters which will behave in a mathematically simple way.

This approach is valid for polycrystalline materials, but the assumption of isotropy is not valid for single crystals or textured materials, e.g., as occurs when drawing a cylindrical cup from a flat circular block (c.f., C6 (Crystallography)).

The microscopic approach and the macroscopic approach are used in parallel. They are equally valid **in their own spheres of application**. In this course, we will look mainly at the macroscopic continuum mechanics approach.

Pictorial summary:

	Microscopic approach	**Macroscopic approach**
Elasticity		$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$
Plasticity	Dislocation motion	Continuum plasticity
	physics ←	engineering →

barrier

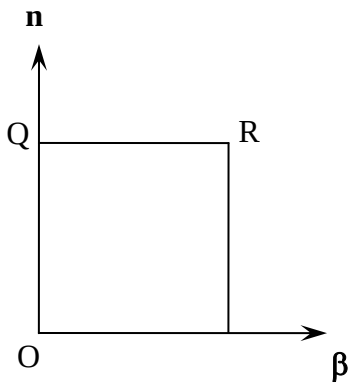
To set the scene for this course on plasticity, we first look at the microscopic picture of plastic flow in more detail.

Microscopically, plastic flow is due to slip on a plane, or a set of planes, within a crystal. In order to understand why some crystals will flow plastically more readily than others, we need to know how many slip systems in different structures cause shape changes in a crystal, e.g., following Kelly, Groves and Kidd, we need to answer the question

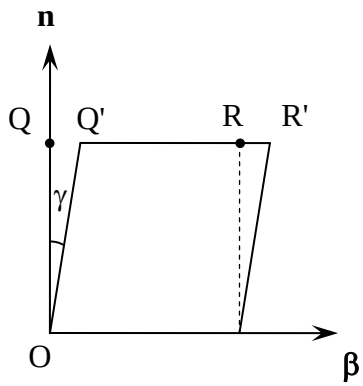
How does slip on a particular crystal system relate to the strain tensor produced?

Suppose we first consider simple shear of a crystal through a small angle γ on a slip plane defined by a unit vector **n** normal to it in a slip direction lying in the slip plane defined by the unit vector $\boldsymbol{\beta}$.

In the diagram below, the vector **OQ** initially parallel to **n** is sheared to become a vector **OQ**' after the simple shear has taken place, and likewise the vector **OR** lying in the plane containing **n** and $\boldsymbol{\beta}$ is sheared to a vector **OR**':



(a) before shear



(b) after shear

From this diagram it is apparent that the displacement **QQ**' is the same magnitude as the displacement **RR**'.

Clearly, all vectors **r** in the plane containing **n** and $\boldsymbol{\beta}$ whose projected length along **n** is OQ suffer the same displacement. Such vectors will have a component along the unit vector **n** of magnitude **r.n**.

Also, from the diagram,

$$\frac{|QQ'|}{|OQ|} = \tan\gamma \cong \gamma \quad \text{if } \gamma \text{ is small (with } \gamma \text{ measured in radians).}$$

Hence, if γ is small, $\mathbf{QQ}' = \gamma(\mathbf{r.n})\boldsymbol{\beta}$.

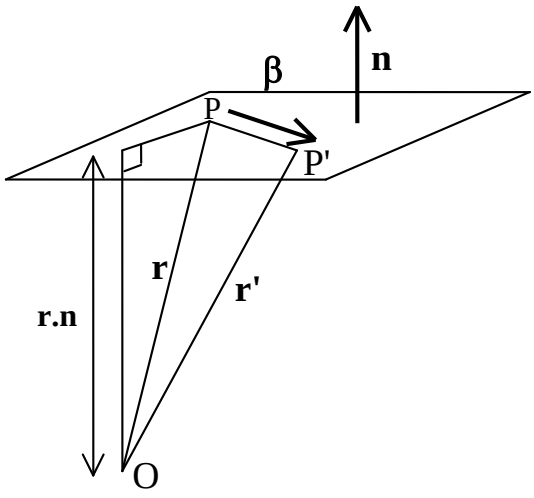
In general, we need to consider the situation in 3D where **r** can also have a component along a vector normal to both **n** and $\boldsymbol{\beta}$. Components along the vector normal to both **n** and $\boldsymbol{\beta}$ are unaffected by the shear process.

Thus, consideration of the general case on page 7 shows that the important features when considering the effect of a shear process are γ and the component of **r** parallel to **n**, which is still **r.n**.

Hence, as quoted on page 7, the result for glide on a general plane in a general direction is that the amount of shear is

$$\mathbf{PP}' = \gamma(\mathbf{r.n})\boldsymbol{\beta}$$

Strain tensor for glide on a general plane in a general direction



Define unit vectors w.r.t. an orthonormal axis system:

n normal to slip plane

β in slip direction

Slip moves point P to P'

where $\mathbf{PP'} = \mathbf{r'} - \mathbf{r} = \gamma(\mathbf{r}.\mathbf{n})\,\boldsymbol{\beta}$

and γ is the angle of shear due to glide

n has components $n_1\ n_2\ n_3$
β has components $\beta_1\ \beta_2\ \beta_3$
r has components $x_1\ x_2\ x_3$

For small γ the relative displacement tensor e_{ij} is obtained by differentiating the displacements, e.g.,

$$e_{11} = \frac{\partial u_1}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\mathbf{r'}-\mathbf{r}\right)_1 = \frac{\partial}{\partial x_1}\gamma\left(\mathbf{r}\cdot\mathbf{n}\right)\beta_1 = \gamma\beta_1\frac{\partial}{\partial x_1}(x_1n_1 + x_2n_2 + x_3n_3) = \gamma n_1\beta_1$$

Similarly for all e_{ij}:

$$e_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\mathbf{r'}-\mathbf{r}\right)_i = \frac{\partial}{\partial x_j}\gamma(x_1n_1 + x_2n_2 + x_3n_3)\beta_i = \gamma n_j\beta_i$$

$$e_{ij} = \gamma\begin{pmatrix} n_1\beta_1 & n_2\beta_1 & n_3\beta_1 \\ n_1\beta_2 & n_2\beta_2 & n_3\beta_2 \\ n_1\beta_3 & n_2\beta_3 & n_3\beta_3 \end{pmatrix}$$

This relative displacement tensor can be separated into a symmetrical strain tensor ε_{ij} and an antisymmetric rotation tensor ω_{ij}, where

$$\varepsilon_{ij} = \gamma\begin{pmatrix} n_1\beta_1 & \frac{1}{2}(n_2\beta_1 + n_1\beta_2) & \frac{1}{2}(n_3\beta_1 + n_1\beta_3) \\ \frac{1}{2}(n_1\beta_2 + n_2\beta_1) & n_2\beta_2 & \frac{1}{2}(n_3\beta_2 + n_2\beta_3) \\ \frac{1}{2}(n_1\beta_3 + n_3\beta_1) & \frac{1}{2}(n_2\beta_3 + n_3\beta_2) & n_3\beta_3 \end{pmatrix}$$

and

$$\omega_{ij} = \gamma\begin{pmatrix} 0 & \frac{1}{2}(n_2\beta_1 - n_1\beta_2) & \frac{1}{2}(n_3\beta_1 - n_1\beta_3) \\ \frac{1}{2}(n_1\beta_2 - n_2\beta_1) & 0 & \frac{1}{2}(n_3\beta_2 - n_2\beta_3) \\ \frac{1}{2}(n_1\beta_3 - n_3\beta_1) & \frac{1}{2}(n_2\beta_3 - n_3\beta_2) & 0 \end{pmatrix}$$

The strain and rotation tensors derived on page 7 are actually less fearsome than they look, particularly when we apply these tensors to 'sensible' axes.

For example, a measure of the volume change is

$$\frac{\Delta V}{V} = \varepsilon_{ii} \equiv \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \Delta$$

[Remember that from C3, the trace of a strain matrix is an invariant, i.e., it does not depend on the axis system used.]

Hence,

$$\Delta = \varepsilon_{ii} = \gamma \left(n_1\beta_1 + n_2\beta_2 + n_3\beta_3 \right) = \gamma\, \mathbf{n}.\boldsymbol{\beta}$$

Since $\boldsymbol{\beta}$ lies in $\mathbf{n}$, it follows that $\mathbf{n}.\boldsymbol{\beta} = 0$ and so $\Delta = 0$.

Therefore, **slip is a macroscopic shape change with no change in volume**. This fits in well with our ideas of dislocation motion – we should not expect a volume change during glide.

Independent slip systems

Formal definition:

A slip system is independent of others if the pure strain ε_{ij} produced by it cannot be produced by a suitable combination of slip on other systems.

Different crystal structures have different slip systems, as shown on page 9. The number of independent slip systems shown can be seen to range from two to five. This number has implications for plastic flow.

Independent glide systems for different crystal structures

Cubic

Slip systems	No. of physically distinct slip systems	No. of independent slip systems	Crystal structure
$<1\bar{1}0>\{111\}$	12	5	c.c.p. metals
$<1\bar{1}1>\{110\}$	12	5	b.c.c. metals
$<1\bar{1}0>\{110\}$	6	2	NaCl structure
<001> {110}	6	3	CsCl structure

Hexagonal close packed metals

Slip systems	No. of physically distinct slip systems	No. of independent slip systems
$<11\bar{2}0>\{0001\}$	3	2
$<11\bar{2}0>\{1\bar{1}00\}$	3	2
$<11\bar{2}0>\{1\bar{1}01\}$	6	4
$<11\bar{2}3>\{11\bar{2}\bar{2}\}$	6	5

For further examples of slip systems see Kelly, Groves and Kidd, pages 188–189.

Independent glide systems in crystals with the NaCl structure: proof of two independent slip systems

At room temperature the slip systems are $\{110\}<1\bar{1}0>$ in crystals with the NaCl structure.

There are six distinct planes of the type {110}. Within each plane there is only one distinct $<1\bar{1}0>$ direction. Hence, there are six physically distinct slip systems.

It is convenient to label the six slip systems A – F for simplicity:

Label	Slip system
A	$(110)[1\bar{1}0]$
B	$(1\bar{1}0)[110]$
C	$(011)[0\bar{1}1]$

Label	Slip system
D	$(0\bar{1}1)[011]$
E	$(101)[10\bar{1}]$
F	$(10\bar{1})[101]$

Looking at slip system A, the strain tensor produced by shearing an angle γ is

$$\boldsymbol{\varepsilon}_{\mathrm{A}} = \frac{\gamma}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

since $\mathbf{n} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]$ and $\boldsymbol{\beta} = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]$ (choosing our orthonormal set of axes to be parallel to the crystal axes).

Evaluating the other five strain tensors in a similar manner, we find:

$$\boldsymbol{\varepsilon}_{\mathrm{B}} = \frac{\gamma}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{C}} = \frac{\gamma}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

$$\boldsymbol{\varepsilon}_{\mathrm{D}} = \frac{\gamma}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{E}} = \frac{\gamma}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{F}} = \frac{\gamma}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It is immediately apparent that

$$\boldsymbol{\varepsilon}_A = \boldsymbol{\varepsilon}_B$$

$$\boldsymbol{\varepsilon}_C = \boldsymbol{\varepsilon}_D$$

$$\boldsymbol{\varepsilon}_E = \boldsymbol{\varepsilon}_F$$

so that there are at most three independent slip systems. Of course the rotation tensors $\boldsymbol{\omega}_A$ and $\boldsymbol{\omega}_B$ are different. Likewise, $\boldsymbol{\omega}_C$ and $\boldsymbol{\omega}_D$ are different from one another, as are $\boldsymbol{\omega}_E$ and $\boldsymbol{\omega}_F$.

However, an examination of $\boldsymbol{\varepsilon}_E$ shows that it can be produced by a linear combination of $\boldsymbol{\varepsilon}_A$ and $\boldsymbol{\varepsilon}_C$:

$$\boldsymbol{\varepsilon}_E = \boldsymbol{\varepsilon}_A + \boldsymbol{\varepsilon}_C$$

i.e., in words, shear of an amount γ on the slip system $(110)[1\bar{1}0]$ combined with shear of the same amount γ on $(011)[0\bar{1}1]$ together produce a shape strain equivalent to shear of an amount γ on the slip system $(101)[10\bar{1}]$.

Hence there are only two independent slip systems in crystals with the NaCl structure, e.g., A and C.

Independent glide systems in c.c.p. metals: proof of five independent slip systems

There are 12 slip systems of the form $<1\bar{1}0>\{111\}$. Considering each of these in turn, we can choose to look at slip directions within a given slip plane, and the strain tensors they produce.

Example: slip along $[1\bar{1}0]$ on the (111) slip plane:

If the amount of slip is of magnitude γ, where γ is the angle of shear due to glide,

$$\boldsymbol{\varepsilon} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

since $\mathbf{n} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$ and $\boldsymbol{\beta} = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]$.

We can therefore determine the 12 strain tensors produced by the 12 slip systems for slip of magnitude γ, designating the slip systems A – L.

Label	Slip system
A	$[1\bar{1}0](111)$
B	$[0\bar{1}1](111)$
C	$[\bar{1}01](111)$
D	$[1\bar{1}0](11\bar{1})$
E	$[011](11\bar{1})$
F	$[101](11\bar{1})$

Label	Slip system
G	$[0\bar{1}\bar{1}](1\bar{1}1)$
H	$[110](1\bar{1}1)$
I	$[10\bar{1}](1\bar{1}1)$
J	$[110](\bar{1}11)$
K	$[\bar{1}0\bar{1}](\bar{1}11)$
L	$[0\bar{1}1](\bar{1}11)$

The strain tensors are therefore the following matrices:

$$\boldsymbol{\varepsilon}_{\mathrm{A}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{B}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & -2 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{C}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

$$\boldsymbol{\varepsilon}_{\mathrm{D}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 2 & 0 & -1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{E}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{F}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\boldsymbol{\varepsilon}_{\mathrm{G}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & -2 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{H}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{I}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\boldsymbol{\varepsilon}_{\mathrm{J}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} -2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{K}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -2 \end{pmatrix}; \quad \boldsymbol{\varepsilon}_{\mathrm{L}} = \frac{1}{2\sqrt{6}}\gamma\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

It is readily apparent that there can only be two independent slip systems on each {111} slip plane, so that, for example, in this table

$$\boldsymbol{\varepsilon}_{\mathrm{C}} = \boldsymbol{\varepsilon}_{\mathrm{A}} + \boldsymbol{\varepsilon}_{\mathrm{B}}$$

$$\boldsymbol{\varepsilon}_{\mathrm{F}} = \boldsymbol{\varepsilon}_{\mathrm{D}} + \boldsymbol{\varepsilon}_{\mathrm{E}}$$

$$\boldsymbol{\varepsilon}_{\mathrm{I}} = \boldsymbol{\varepsilon}_{\mathrm{G}} + \boldsymbol{\varepsilon}_{\mathrm{H}}$$

$$\boldsymbol{\varepsilon}_{\mathrm{L}} = \boldsymbol{\varepsilon}_{\mathrm{J}} + \boldsymbol{\varepsilon}_{\mathrm{K}}$$

and so we are not totally unrestricted in the choice of the slip systems that we identify as candidate independent slip systems for c.c.p. crystals. Therefore, we need only look at slip systems A, B, D, E, G, H, J and K.

If out of these we choose our independent slip systems to be A, B, D, E, and G, the strain tensors of the other seven slip systems can all be written in terms of $\boldsymbol{\varepsilon}_{\mathrm{A}}, \boldsymbol{\varepsilon}_{\mathrm{B}}, \boldsymbol{\varepsilon}_{\mathrm{D}}, \boldsymbol{\varepsilon}_{\mathrm{E}}$ and $\boldsymbol{\varepsilon}_{\mathrm{G}}$, demonstrating that there are five independent slip systems for c.c.p. crystals.

[An examination of the forms of the strain tensor for A, B, D, E and G shows that they are all independent of one another. Thus, for example, the strain tensor for G cannot be constructed from a suitable combination of the strain tensors of A, B, D and E to reduce the number of independent slip systems still further.]

Thus,

$$\boldsymbol{\varepsilon}_C = \boldsymbol{\varepsilon}_A + \boldsymbol{\varepsilon}_B$$

$$\boldsymbol{\varepsilon}_F = \boldsymbol{\varepsilon}_D + \boldsymbol{\varepsilon}_E$$

$$\boldsymbol{\varepsilon}_H = \boldsymbol{\varepsilon}_D + \boldsymbol{\varepsilon}_E - \boldsymbol{\varepsilon}_B$$

$$\boldsymbol{\varepsilon}_I = \boldsymbol{\varepsilon}_D + \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_G - \boldsymbol{\varepsilon}_B$$

$$\boldsymbol{\varepsilon}_J = \boldsymbol{\varepsilon}_E - \boldsymbol{\varepsilon}_A - \boldsymbol{\varepsilon}_B$$

$$\boldsymbol{\varepsilon}_K = \boldsymbol{\varepsilon}_A + \boldsymbol{\varepsilon}_G$$

$$\boldsymbol{\varepsilon}_L = \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_G - \boldsymbol{\varepsilon}_B$$

It is also immediately apparent from this analysis that there have to be five independent slip systems in b.c.c. crystals.

Plastic flow in crystals

We know that in general there are 6 independent components of a strain tensor because strain tensors have to be symmetric.

However, for plastic flow we also know that the volume change is zero, i.e.,

$$\Delta = \varepsilon_{ii} \equiv \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0$$

Hence, knowing ε_{11} and ε_{22}, we can determine ε_{33}, and so only two of these are needed, not three.

Therefore, for a perfectly general plastic shape change with constant volume, we only have **5** independent strain components.

It therefore follows that to achieve a general plastic shape change by slip, we need 5 independent slip systems. This was first recognised in 1928 by Richard Edler von Mises (see http://en.wikipedia.org/wiki/Richard_von_Mises) in a paper 'Mechanik der plastischen Formänderung von Kristallen' published in *Zeitschrift für Angewandte Mathematik und Mechanik*, **8**, 161–185.

Another general result which follows from the need for 5 independent slip systems is that 5 are **all** we need, and in fact there cannot be more than 5.

Crystals with the NaCl crystal structure have only two independent slip systems. Hence we can infer that such crystals not be ductile in a general loading situation – some strains will not be able to be accommodated by dislocation flow. Hence, we infer that **fracture** will occur to relieve the strains generated. This ties in with our everyday experience – salt crystals fail by fracture and can easily be crushed into many small pieces.

However, instead of a general loading, a specific loading situation is considered, then ductility can occur in crystals with the NaCl structure.

If a single crystal of LiF is compressed along a <100> direction, it will flow. Dropping SiC grit particles onto a (100) surface of a LiF single crystal and examining the surface afterwards by optical microscopy shows evidence for dislocation mobility.

Compressing a single crystal of LiF along <111> will cause it to shatter without any dislocation flow because the Schmid factors of each of the six possible slip systems which operate in LiF will all be zero.

Although the number of independent slip systems can tell us a lot about the plastic behaviour of a material, materials with five independent slip systems can only accommodate a general strain if these slip systems can operate simultaneously in a small volume of the material – there must be **slip flexibility**.

For slip flexibility dislocations must be able to cross-slip easily and slip bands must interpenetrate, so that dislocations on one system are not blocked by dislocations on another system.

Hence the von Mises condition tells us if a material **can** be ductile, but not if it **is**.

Examples of limited ductility

- **CsBr** (CsCl structure)

CsBr has very flexible slip, but since it only has 3 independent slip systems of the form <001> {110}, and so it therefore has limited ductility.

- **MgO** (NaCl structure)

Between room temperature and 350 °C, the slip systems are $<1\bar{1}0>$ {110} and so there are only two independent systems and therefore limited ductility (see the example above for LiF).

Above 350 °C we get a second set of slip systems, <001> {110}, giving in total five independent slip systems, but cross-slip is still very difficult – the material is not ductile.

For $T > 1750$ °C, the slips systems are able to interpenetrate, giving fully ductile behaviour, and so at very high temperatures, MgO behaves likes a c.c.p. metal.

Hexagonal metals (e.g., Reed-Hill, pp. 180–183)

In many hexagonal metals slip occurs on $<11\bar{2}0>\{0001\}$, for which there are only two independent slip systems. This follows immediately from the recognition that $<11\bar{2}0>\{0001\}$ is basal plane slip.

Hexagonal metals can also slip on pyramidal planes, e.g. $<11\bar{2}0>\{1\bar{1}01\}$ and $<11\bar{2}3>\{11\bar{2}\bar{2}\}$. Together with basal plane slip, slip on pyramid planes enables full ductility with five independent slip systems. However, pyramidal slip only becomes easy enough to operate at high *T*.

At low *T*, hexagonal metals such as Mg, Zn and Be tend to have low ductility and so must be hot worked during deformation processing.

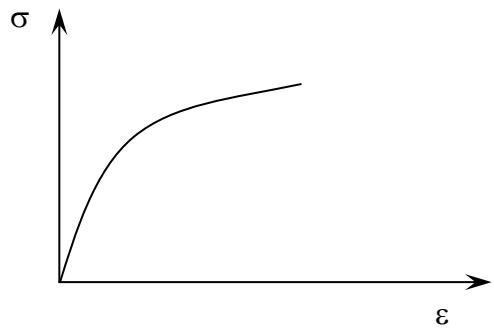
Ti, which is also an h.c.p. metal has reasonable ductility at low *T* because it slips readily on $<11\bar{2}0>\{1\bar{1}00\}$ as well.

Continuum plasticity

We now turn to the macroscopic view of plasticity where we consider materials as homogeneous isotropic media in the first instance with relatively simple constitutive equations.

This theory is vital in engineering design, forming and fabrication operations in metals, e.g., machining, forging and extrusion.

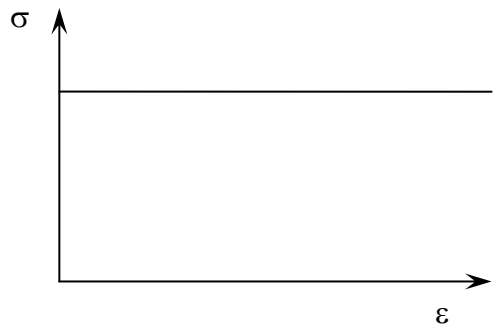
We begin by considering the $\sigma - \varepsilon$ curve of a real polycrystalline material:



This type of stress–strain curve can be approximated in various ways:

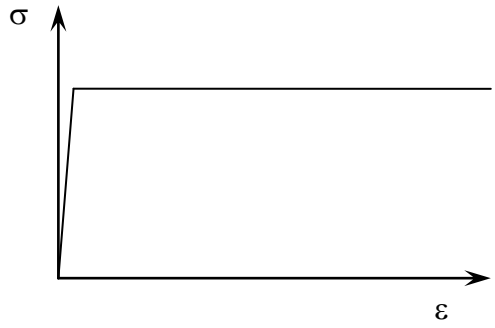
(1) *Rigid – perfect plastic*

In this approximation, the material is regarded as perfectly rigid prior to the onset of plastic behaviour and perfectly plastic after the onset, i.e., a plastic material with no work hardening. Hence the stress–strain curve looks like:



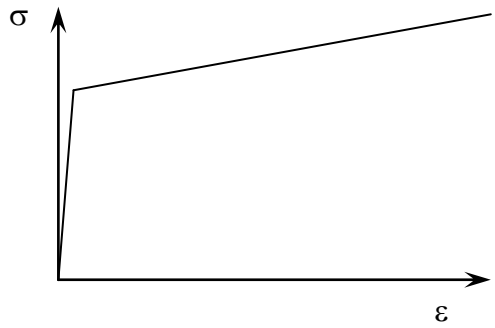
(2) *Linear elastic – perfect plastic*

Here, the material is regarded linear elastic prior to the onset of plastic behaviour and perfectly plastic after the onset, i.e., a plastic material with no work hardening. Hence the stress–strain curve looks like:



(3) *Linear elastic – linear work hardening*

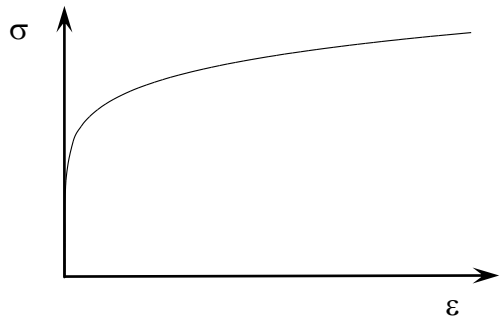
As the name suggests, the stress–strain curve looks like:



There are of course other possibilities. One in particular is to approximate the relationship between true stress and true strain to be of the form

$$\sigma = K\varepsilon^n$$

for a suitable work hardening exponent n:



The most simple of these approximations is that of **rigid – perfect plastic**.

Here, elastic strains are completely ignored. This is justifiable since elastic strains are typically < 1%, whereas plastic strains can be up to 50% or more. This is a very good model to use for metals with very low work-hardening rates, or when either hot working or using low strains.

To construct a theory of plasticity, we need to define some criterion which will tell us when yielding will occur for any stress state, not just uniaxial tension.

We begin with metals.

Empirically, it is observed that hydrostatic pressure has very little or no effect on the yield behaviour of metals. This fits in with our picture of plastic flow in metals due to dislocation motion: such motion can only be influenced by shear stresses. Hydrostatic stresses do not provide shear stresses. Therefore, we should not expect hydrostatic stresses to cause plastic flow in metals.

From C4 (Tensor Properties), the principal stress tensor

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

can be split into a hydrostatic component (dilatational component) and a deviatioric component:

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \sigma_H & 0 & 0 \\ 0 & \sigma_H & 0 \\ 0 & 0 & \sigma_H \end{pmatrix} + \begin{pmatrix} \sigma_1 - \sigma_H & 0 & 0 \\ 0 & \sigma_2 - \sigma_H & 0 \\ 0 & 0 & \sigma_3 - \sigma_H \end{pmatrix}$$

hydrostatic deviatoric

where $\sigma_H = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$.

The hydrostatic stress tensor remains the same irrespective of how the axis system is defined. This can easily be appreciated mathematically, From C3 (Mathematical Methods), the effect of a similarity transformation on the unit 3×3 stress tensor, **I**, is:

$$\mathbf{C}\mathbf{I}\mathbf{C}^{-1} = \mathbf{C}\mathbf{C}^{-1} = \mathbf{I}$$

where **C** is a rotation matrix. The hydrostatic stress tensor is just $\sigma_H \mathbf{I}$.

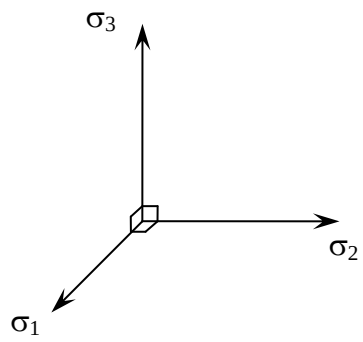
It follows from this elementary consideration that to define when yielding is about to occur, we need to specify some function of the **deviatoric stress tensor** as being satisfied, e.g., when this function (whatever it is) reaches a critical value.

Yield criteria

A yield criterion is ‘An hypothesis defining the limit of elasticity under any possible combination of stresses’.

There are several possible yield criteria. We shall examine two of these in some detail.

First, it is expedient to introduce the concept of **principal stress space** to help our understanding. We define a right-handed orthogonal 3D space with axes σ_1, σ_2 and σ_3:



This is not of course **real space**. The orthogonal axes σ_1, σ_2 and σ_3 axes are not (necessarily) related to orthogonal crystal axes.

Using this construction, **any** stress state can be plotted on this diagram as a **point** in 3D stress space (although the stress state will have to be referred to the principal stresses, and this could well involve determining these for a general stress matrix with six independent components).

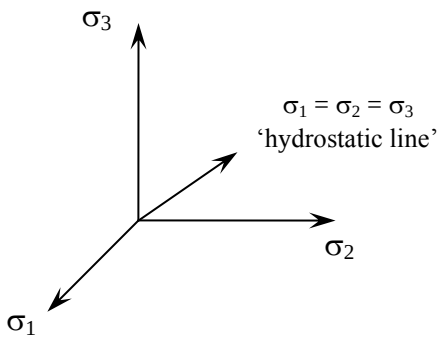
The most simple stress state of uniaxial tension with $\sigma_1 = \sigma$, $\sigma_2 = 0$ and $\sigma_3 = 0$ has a stress tensor of the form

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so this plots as a point on the σ_1 axis at σ.

If σ reaches the value at which yielding occurs in unaxial tension, the yield stress, *Y*, we have reached a critical value of the function of the deviatoric stress tensor at this point in principal stress space.

A purely hydrostatic stress $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ will lie on the line $\sigma_1 = \sigma_2 = \sigma_3$, i.e., the 'vector' [111] referred to the 'vectors' $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ defining principal stress space.



From the recognition that hydrostatic pressure has no effect on the yield behaviour of metals it follows that, for any point on this hydrostatic line, there can be no yielding.

Now, we have already deduced that if $\sigma_1 = Y$, $\sigma_2 = 0$ and $\sigma_3 = 0$, yielding occurs.

Therefore, there must be a surface which surrounds the hydrostatic line and passes through (Y, 0, 0) which is the boundary between elastic and plastic behaviour.

By symmetry, this boundary must also pass through (0, Y, 0) and (0, 0, Y).

If we can find some suitable surface, then that surface defines a yield criterion.

von Mises yield criterion

The most simple shape we could consider is a cylinder of appropriate radius whose axis is along the hydrostatic line, i.e.,

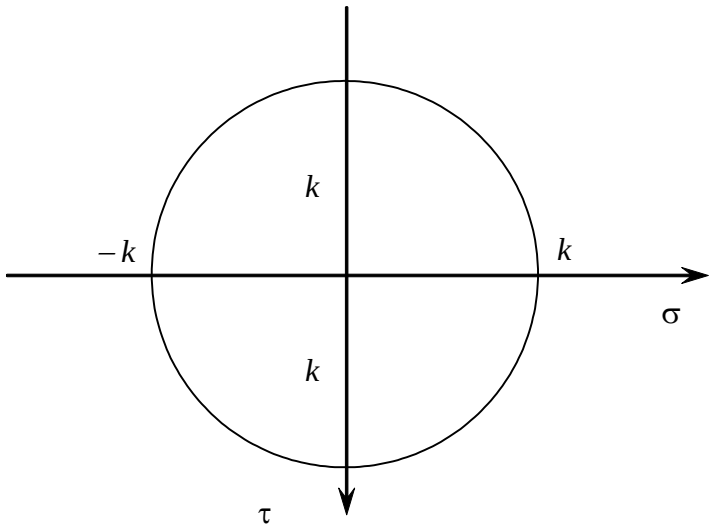
$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = \text{constant}$$

This is the von Mises yield criterion.

Determining the constant is straightforward. Since the cylinder passes through (Y, 0, 0), it follows that the constant is $2Y^2$. Hence the von Mises yield criterion is

$$\boxed{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2}$$

We can also define a yield stress in pure shear. Using Mohr's circle construction for a pure shear yield stress of magnitude k, we have:



Therefore, referred to principal stress space we have $\sigma_1 = k$, $\sigma_2 = -k$ and $\sigma_3 = 0$ when the magnitude of the shear stress is that required to just cause yielding.

Hence, the constant is also $6k^2$. Therefore, using the von Mises yield criterion, the relationship between the uniaxial yield stress, Y, and the shear yield stress, k, is

$$\boxed{Y = \sqrt{3}k}$$

and so the von Mises yield criterion can be written

$$\boxed{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2}$$

Physical interpretation of the von Mises yield criterion

There are a number of ways in which the von Mises yield criterion can be interpreted physically.

If we neglect dilatational changes, the von Mises criterion is very similar to the elastic strain energy density arising from distortion alone:

$$U_{\text{distortional}} = \frac{1}{12G}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}$$

where G is the shear modulus (see, for example, Johnson and Mellor, p. 71).

Therefore, we can regard the von Mises criterion as one where yield occurs when the strain energy density reaches a critical value.

Hence, the von Mises criterion is sometimes called the **distortional energy criterion**.

Alternatively, if we look at normal and shear stresses on {111} octahedral 'planes' with respect to the orthogonal and orthonormal principal stress axes, then we find that the shear stresses on these 'planes' (remember they are unlikely to be crystallographic {111} planes) have a magnitude

$$|\tau_{\text{oct}}| = \frac{1}{3}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}^{1/2}$$

(see Question Sheet 1). Hence, we can interpret the von Mises yield criterion as a **critical octahedral shear stress criterion**.

Invariants of stress tensors and the von Mises yield criterion

In its most general form a stress tensor can be written in the form

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz} \end{pmatrix}$$

Principal stresses σ_1, σ_2 and σ_3 are roots of the determinant equation

$$\left|\sigma_{ij} - \lambda\delta_{ij}\right| = 0$$

where δ_{ij} is the Kronecker delta (δ_{ij} =1 if $i = j$ and 0 otherwise).

Writing this equation out in full we have the cubic equation

$$\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0$$

(Backofen, page 7), where

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = -\sigma_{xx}\sigma_{yy} - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx} + \tau_{xy}^{\ 2} + \tau_{yz}^{\ 2} + \tau_{zx}^{\ 2}$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^{\ 2} - \sigma_{yy}\tau_{zx}^{\ 2} - \sigma_{zz}\tau_{xy}^{\ 2}$$

The coefficients I_1, I_2 and I_3 are invariant, i.e., their values do not change with orientation because the three roots of this cubic equation (the principal stresses) do not.

If the coordinates in which the stress tensor is described is one in which the *x*-, *y*- and *z*-directions are the directions of the principal stresses, it follows that

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = -\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

demonstrating from I_3 that the determinant of a stress tensor is an invariant and from I_1 that the trace of a stress tensor is also an invariant.

The deviatoric stress tensor σ'_{ij} is related to σ_{ij} through the equation

$$\sigma'_{ij} = \sigma_{ij} - \frac{I_1}{3}$$

i.e., it is the original stress tensor with the hydrostatic stress subtracted. Invariants of this deviatoric stress tensor can also be defined as roots of the determinant equation

$$\left|\sigma'_{ij} - \lambda\delta_{ij}\right| = 0$$

Writing out in full this becomes

$$\lambda^3 - J_2\lambda - J_3 = 0$$

The coefficient of λ^2 is zero here because if follows from the definition of the deviatoric stress tensor that its trace is zero.

If the coordinates in which the deviatoric stress tensor is described is one in which the *x*-, *y*- and *z*-directions are the directions of the principal stresses, the quadratic invariant, J_2, becomes

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

If, as is the case for metals, yielding is not affected by the magnitude of the mean normal stress (i.e., the value of hydrostatic stress), then **any yield criterion has only to involve the components of the deviatoric stress tensor**. A simple yield criterion under these circumstances is that one of the invariants of the deviatoric stress tensor should be a constant at yielding. Consideration of the form of J_2 shows that the von Mises yield condition is equivalent to the statement that, at yielding,

$$J_2 = \text{constant}$$

The value of this constant can be established from the condition that at yield in uniaxial tension, the principal stresses $\sigma_1 = Y$, $\sigma_2 = 0$ and $\sigma_3 = 0$, whence

$$J_2 = \frac{Y^2}{3}$$

and so this yield condition is simply rewritten in the more familiar form

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2$$

A mathematically simpler criterion is the **Tresca criterion**, advocated in 1864 by Henri Édouard Tresca, a French mechanical engineer (see http://en.wikipedia.org/wiki/Henri_Tresca):

'Yield occurs when the maximum shear stress reaches a critical value'

Suppose we have a principal stress state where $\sigma_1 > \sigma_2 > \sigma_3$. The criterion reduces to the statement that yield occurs when

$$(\sigma_1 - \sigma_3) = \text{constant}$$

So if yielding occurs when $\sigma_1 = Y$, $\sigma_2 = 0$ and $\sigma_3 = 0$ (uniaxial tension), it follows that

$$(\sigma_1 - \sigma_3) = Y$$

If yield occurs in pure shear when $\sigma_1 = k$, $\sigma_2 = 0$ and $\sigma_3 = -k$, it follows that

$$(\sigma_1 - \sigma_3) = 2k$$

Hence the Tresca criterion for $\sigma_1 > \sigma_2 > \sigma_3$ is

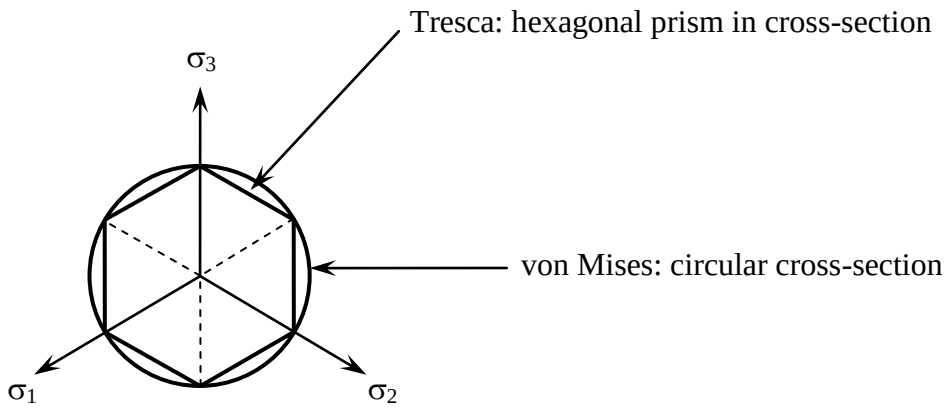
$$\boxed{(\sigma_1 - \sigma_3) = Y = 2k}$$

clearly showing that using the Tresca yield criterion, the relationship between the uniaxial yield stress, Y, and the shear yield stress, k, is

$$\boxed{Y = 2k}$$

which compares with $Y = \sqrt{3}k$ for the von Mises criterion.

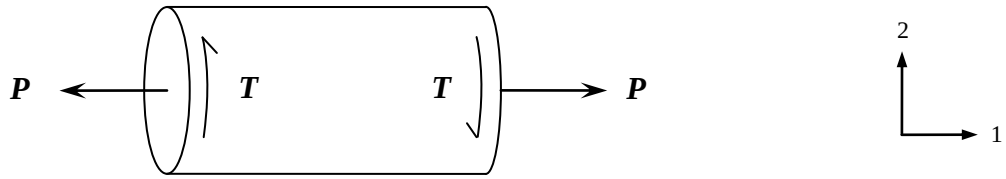
Viewed down the hydrostatic line, the von Mises yield criterion and the Tresca yield criterion plot as:



Experimental test of the von Mises and Tresca yield criteria

To determine which of these two yield criteria is most appropriate to describe the yield behaviour of metallic materials, we need to be able to subject an object to combined stress states. The classic experiment was first performed by Taylor and Quinney in 1931 and has been repeated many times since.

In such an experiment, use is made of a thin-walled tube subjected to a combined torque, *T*, and axial load, *P*:



Then, with respect to the axis system ‘1’ along the axis and ‘2’ around the cylinder, the stress tensor takes the form

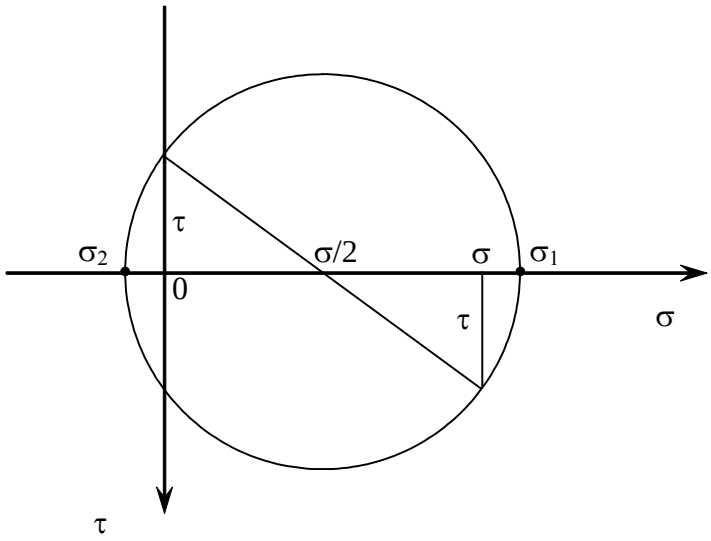
$$\begin{pmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$\sigma = \frac{P}{2\pi r t} \quad \text{and} \quad \tau = \frac{T}{2\pi r^2 t}$$

for a thin-walled tube of radius *r* and wall thickness *t*.

For this stress state we can use Mohr's circle to determine the principal stresses σ_1 and σ_2. $\sigma_3 = 0$ by inspection.



From Mohr's circle it is apparent that

$$\sigma_1 = \frac{\sigma}{2} + \left(\tau^2 + \frac{\sigma^2}{4}\right)^{1/2} \quad \text{and} \quad \sigma_2 = \frac{\sigma}{2} - \left(\tau^2 + \frac{\sigma^2}{4}\right)^{1/2}$$

Hence, the maximum principal stress is σ_1 and the minimum principal stress is σ_2.

Tresca criterion

This reduces to

$$(\sigma_1 - \sigma_2) = Y$$

Hence,

$$2\left(\tau^2 + \frac{\sigma^2}{4}\right)^{1/2} = Y$$

i.e.,

$$\tau^2 + \frac{\sigma^2}{4} = \frac{Y^2}{4}$$

which can be rearranged in the form

$$\boxed{\sigma^2 + 4\tau^2 = Y^2}$$

von Mises criterion

This becomes

$$4\left(\tau^2 + \frac{\sigma^2}{4}\right) + 2\frac{\sigma^2}{4} + 2\left(\tau^2 + \frac{\sigma^2}{4}\right) = 2Y^2$$

which can be rearranged after some straightforward algebra to become

$$\boxed{\sigma^2 + 3\tau^2 = Y^2}$$

The relevant conditions to satisfy the two criteria are therefore

$$\sigma^2 + 4\tau^2 = Y^2 \qquad \text{Tresca}$$

$$\sigma^2 + 3\tau^2 = Y^2 \qquad \text{von Mises}$$

These can be rearranged in the forms

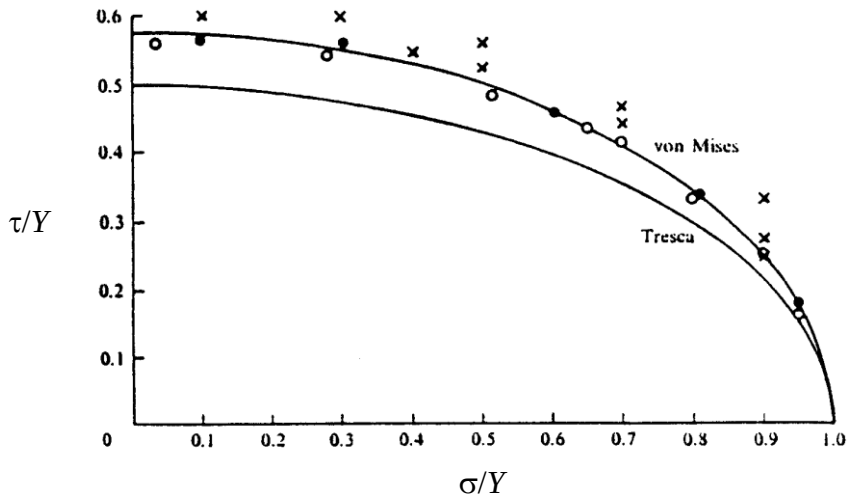
$$\frac{\sigma^2}{Y^2} + \frac{\tau^2}{(Y/2)^2} = 1 \qquad \text{Tresca}$$

$$\frac{\sigma^2}{Y^2} + \frac{\tau^2}{\left(Y/\sqrt{3}\right)^2} = 1 \qquad \text{von Mises}$$

i.e., ellipses in $\tau - \sigma$ stress space.

Experimentally, we can vary the load P and the torque T for a thin-walled tube. Hence we can determine the yield stress for different values of P and T (and therefore different values of σ and τ) and compare with the predictions of von Mises and Tresca.

The experimental data collected by Taylor and Quinney below suggested that the von Mises yield criterion fitted the data better than the more conservative Tresca yield criterion.



○ copper, • aluminium and × steel.

Test of yield criteria for metals using thin-walled tubes subjected to combined tensile and shear stresses (G.I. Taylor and H. Quinney, *Phil. Trans. Roy. Soc. Lond.* A**230,** 323 (1931)).

Note that the experiments have to be done with care. Precautions have to be taken against **anisotropy**. If we make a tube by drawing, it will have considerable texture in the drawing direction and will need careful annealing before use.

A test for anisotropy is to measure the internal volume change during deformation by filling the cylinder with a liquid. If the tube is isotropic, there should be no internal volume change during plastic deformation.

Yield criteria for non-metals

Ceramics, when they deform plastically, usually obey the von Mises or the Tresca criterion.

However, other materials such as polymers, concrete, soils and granular materials display yield criteria which are **not** independent of hydrostatic pressure. [Metallic glasses also show a weak hydrostatic dependence on their yield criteria – see, for example, J.J. Lewandowski and P. Lowhaphandu, *Phil. Mag.* A, **82**, 3427 (2002).]

Empirically, it is seen in such materials that, as hydrostatic **pressure** is increased, the yield stress increases, so clearly we do not expect a yield criterion based solely on the deviatoric component of stress to be valid.

The first attempt to produce a yield criterion incorporating the effect of pressure was by Charles-Augustin de Coulomb (1773). This was applied to the shear strength of a soil and also the fracture of building stone in compression.

The Coulomb criterion states that:

Failure occurs when the shear strength τ *on any plane reaches a critical value,* τ_c*, which varies linearly with the stress normal to that plane*

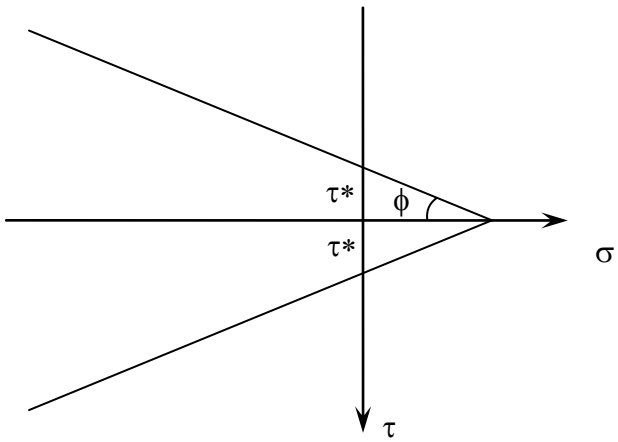
Mathematically the equation describing this is

$$\boxed{\tau_c = \tau^* - \sigma_n \tan\phi}$$

where σ_n is a positive normal stress on the plane of failure (so that a compressive stress would be negative), τ^* is a material parameter and ϕ is an angle of shearing resistance.

Note that $\tan\phi$ is not a 'coefficient of friction', although it is often referred to as such.

An example of the failure locus is shown below in $\tau - \sigma$ space.

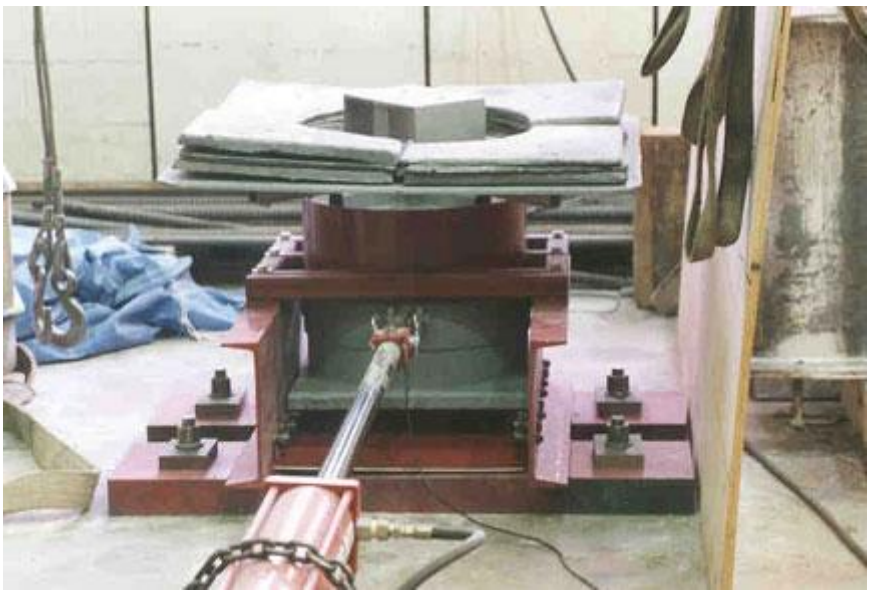


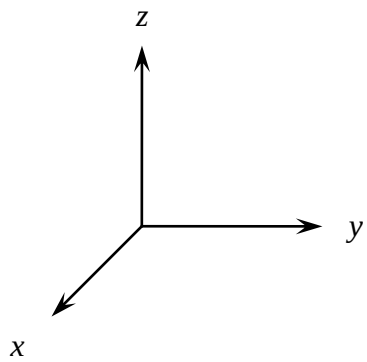
Typically for soils $\tan\phi \approx 0.5 - 0.6$. For metallic glasses, $\tan\phi \approx 0.025$.

From this failure locus it is apparent that, as a soil is compressed, the shear stress required to cause failure increases.

A simple test which demonstrates this is the direct shear test, in which a sample of soil is placed in a stout cylindrical or square-based vessel which can be split in half horizontally. A compressive weight is applied to the top of the sample and a shear stress is applied to the vessel until failure occurs in the soil on the horizontal plane between the two halves of the vessel. Different magnitudes of compressive weights give rise to different values of shear stress required to cause failure.

An example of a large scale direct shear test taken from a recent 2004 U.S. Department of Transportation report is shown below, in which the two halves of the split cylinder are readily apparent.





The test has a number of limitations, not the least of which is the assumption that the intermediate principal stress component lies between the maximum and minimum principal stresses generated by the test, i.e., the test supposes that the stress tensor is of the form

$$\begin{pmatrix} -\sigma_x & 0 & \tau \\ 0 & -\sigma_x & 0 \\ \tau & 0 & -\sigma \end{pmatrix}$$

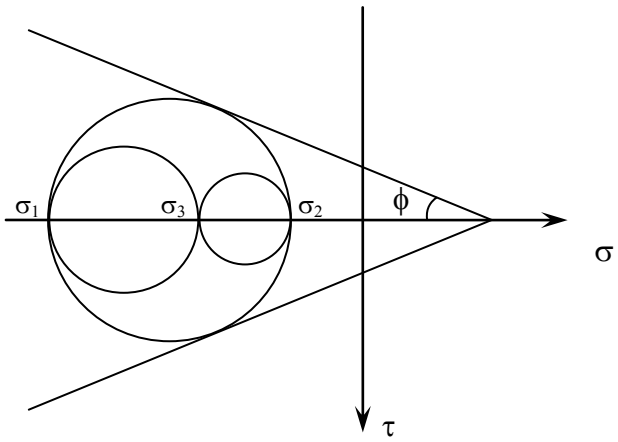
where in the above picture axis '*x*' is out of the plane of the paper, axis '*y*' is horizontal to the right and axis '*z*' is vertical. The tensile stresses $-\sigma_x$ and $-\sigma$ are both compressive to hold the soil in place, and it is presumed that the compressive stress in the *x*-*y* plane is isotropic before the imposition of a shear stress in the *x*-*z* plane.

With respect to principal stresses, we then have

$$\sigma_1 = -\frac{(\sigma+\sigma_x)}{2} - \left(\tau^2 + \frac{(\sigma-\sigma_x)^2}{4}\right)^{1/2}, \quad \sigma_2 = -\frac{(\sigma+\sigma_x)}{2} + \left(\tau^2 + \frac{(\sigma-\sigma_x)^2}{4}\right)^{1/2} \quad \text{and} \quad \sigma_3 = -\sigma_x$$

and we require $\sigma_1 < \sigma_3 < \sigma_2$.

Hence, in terms of a Mohr's circle analysis, we require a state of stress that can be represented in the form



and failure here is determined by $|\sigma_1 - \sigma_2|$, and is independent of σ_3.

Therefore, this failure criterion is a variant or modification of the Tresca failure criterion.

Coulomb's failure model is widely used for soils and has also been used to analyse the yield behaviour of metallic glasses in bulk and strip form.

A better model for **polymers** is to assume that the shear stress required to cause failure is a function of the hydrostatic pressure, *P*, i.e.,

$$k = f(P)$$

such as

$$k = k_0 + \mu P = k_0 - \mu\sigma_H$$

where μ is a dimensionless pressure coefficient and

$$P = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = -\sigma_H$$

where σ_1, σ_2 and σ_3 are the principal stresses.

If we do this, then applying this to the von Mises and Tresca yield criteria, we obtain:

Modified von Mises criterion:

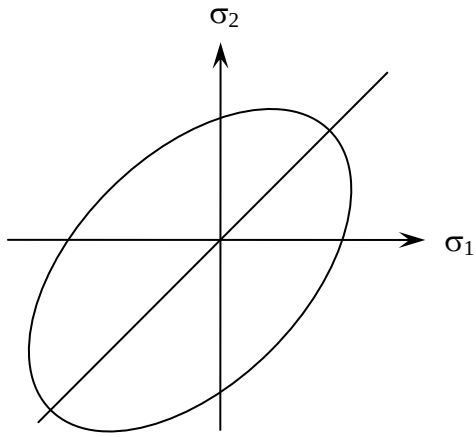
shape defined by a circular sectional cone with its axis along $\sigma_1 = \sigma_2 = \sigma_3$. [See the Appendix].

Modified Tresca criterion:

shape defined by an hexagonal pyramid with its axis along $\sigma_1 = \sigma_2 = \sigma_3$.

As for metals, it is found for polymers that the (modified) von Mises yield criterion works well.

The form of the yield surface for the modified von Mises yield criterion in plane stress, i.e., with $\sigma_3 = 0$, is an ellipse whose centre is displaced away from (0,0) in σ_1-σ_2 space along the $[\bar{1}\bar{1}]$ direction:

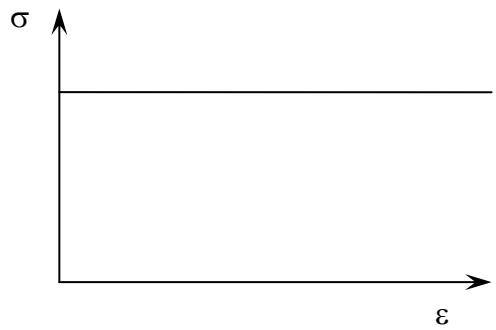


Predictions of plastic strains

Once the yield criterion is satisfied, we can no longer expect to use the equations of elasticity – we must develop a theory to predict **plastic** strains from the imposed stresses.

In general there will be both **plastic** (irrecoverable) and **elastic** (recoverable) strains.

However, we can ignore the elastic strains, assuming that the plastic strains dominate $\Rightarrow$ we can therefore treat the material as **rigid – perfect plastic**:



How do we then relate stress and strain?

Since plasticity is a form of flow, we can relate the strain rate, $d\varepsilon/dt$, to stress, σ.

Plastic flow is similar to fluid flow, except that any rate of flow (strain rate) can occur for the same yield stress.

From symmetry, we can show the following axiom:

In an isotropic body the principal axes of stress and strain rate coincide

i.e., 'it goes the way you push it'.

The behaviour is best described by the Lévy-Mises equations.

With respect to principal axes, the relationships between strain rate and stress take the form

$$\frac{\dot{\varepsilon}_1}{\sigma_1'} = \frac{\dot{\varepsilon}_2}{\sigma_2'} = \frac{\dot{\varepsilon}_3}{\sigma_3'}$$

where $\dot{\varepsilon}_i = \dfrac{d\varepsilon}{dt}$ (i = 1,3) is the normal strain rate parallel to the i^{th} axis.

and σ_i' is the deviatoric component of the normal stress parallel to the i^{th} axis.

Now, $\sigma_1' = \sigma_1 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{2}{3}\left(\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right)$, etc.

If we consider small intervals of time δt and call the resultant changes in strain $\delta\varepsilon_1$, $\delta\varepsilon_2$ and $\delta\varepsilon_3$, it follows that

$$\frac{\delta\varepsilon_1}{\left(\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right)} = \frac{\delta\varepsilon_2}{\left(\sigma_2 - \frac{1}{2}(\sigma_3 + \sigma_1)\right)} = \frac{\delta\varepsilon_3}{\left(\sigma_3 - \frac{1}{2}(\sigma_1 + \sigma_2)\right)}$$

and these are known as the **Lévy-Mises equations**.

Since $\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ is an invariant of the stress tensor, it also turns out that these equations apply even if the stresses and strains are not referred to principal axes, so that

$$\frac{\delta\varepsilon_{11}}{\left(\sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33})\right)} = \frac{\delta\varepsilon_{22}}{\left(\sigma_{22} - \frac{1}{2}(\sigma_{33} + \sigma_{11})\right)} = \frac{\delta\varepsilon_{33}}{\left(\sigma_{33} - \frac{1}{2}(\sigma_{11} + \sigma_{22})\right)}$$

for a general stress tensor and plastic strain increments $\delta\varepsilon_{11}$, $\delta\varepsilon_{22}$ and $\delta\varepsilon_{33}$.

The Lévy-Mises equations are similar to equations that you will recognise from elastic behaviour:

$$E\varepsilon_1 = \sigma_1 - \nu(\sigma_2 + \sigma_3), \text{ etc.}$$

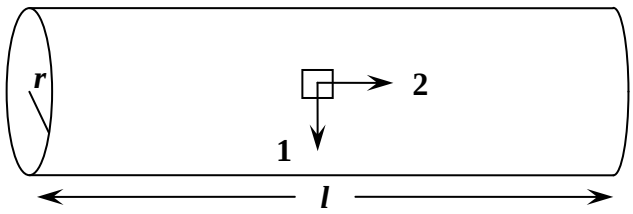
Recalling that for an isotropic material whose Poisson's ratio, ν, is 0.5, there is no volume change (i.e., as happens in plastic flow), then for $\nu = 0.5$ we have:

$$\frac{1}{E} = \frac{\varepsilon_1}{\left(\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right)} = \frac{\varepsilon_2}{\left(\sigma_2 - \frac{1}{2}(\sigma_3 + \sigma_1)\right)} = \frac{\varepsilon_3}{\left(\sigma_3 - \frac{1}{2}(\sigma_1 + \sigma_2)\right)}$$

but here the strains ε are **not** increments of strain.

Example of the analysis of plastic flow

Expansion of a thin-walled cylinder by internal pressure.



This geometry is commonly used for pressure vessels. To design against catastrophic failure, we wish them to flow plastically before fracture (i.e., 'leak before fracture') when subjected to unexpectedly high stresses.

Defining axes 1 and 2 as shown, it is apparent that axis '3' perpendicular to these two will be along the cylinder radius through a small element of interest.

[By symmetry, the principal axes are circumferential, longitudinal and radial.]

Since we are dealing with a thin-walled cylinder, $\sigma_3 = 0$, and so we have plane stress conditions in 1-2 space.

Suppose the internal pressure is P and the wall thickness is t.

It follows from C4 (Tensor Properties) that

$$\sigma_1 = \frac{Pr}{t} \text{ and } \sigma_2 = \frac{Pr}{2t}$$

If we now apply the von Mises yield condition, we find that at yield

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6k^2$$

Now here we have $\sigma_1 = 2\sigma_2$, $\sigma_3 = 0$, and so

$$(2\sigma_2 - \sigma_2)^2 + \sigma_2{}^2 + (2\sigma_2)^2 = 6k^2$$

i.e., at yield

$$\sigma_2 = k$$

$$\sigma_1 = 2k$$

and so the yield pressure, P_{yield}, is given by the equation

$$P_{\text{yield}} = \frac{2tk}{r}$$

and it turns out that in terms of k, this is the **same** yield stress that is predicted on the Tresca yield criterion.

We are now in a position to determine the deformation once yielding has occurred:

Using the Lévy-Mises equations, we have:

$$\frac{\delta\varepsilon_1}{\left(\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right)} = \frac{\delta\varepsilon_2}{\left(\sigma_2 - \frac{1}{2}(\sigma_3 + \sigma_1)\right)} = \frac{\delta\varepsilon_3}{\left(\sigma_3 - \frac{1}{2}(\sigma_1 + \sigma_2)\right)}$$

Now, $\sigma_1 = 2\sigma_2$ and $\sigma_3 = 0$, so we have

$$\frac{\delta\varepsilon_1}{\frac{3}{2}\sigma_2} = \frac{\delta\varepsilon_2}{0} = \frac{\delta\varepsilon_3}{-\frac{3}{2}\sigma_2}$$

from which it is apparent that $\delta\varepsilon_1 = -\delta\varepsilon_3$.

What about $\delta\varepsilon_2$? Suppose that there was a perturbation in the stresses so that $\sigma_1 = 2\sigma_2 - \delta\sigma$ for some incrementally small stress $\delta\sigma$, keeping σ_2 unchanged, and also keeping $\sigma_3 = 0$. Under these circumstances, the Lévy-Mises equations would become

$$\frac{\delta\varepsilon_1}{\frac{3}{2}\sigma_2 - \delta\sigma} = \frac{\delta\varepsilon_2}{\frac{1}{2}\delta\sigma} = \frac{\delta\varepsilon_3}{-\frac{3}{2}\sigma_2 - \frac{\delta\sigma}{2}}$$

and so under these circumstances

$$\frac{\delta\varepsilon_3}{\delta\varepsilon_1} = \frac{-\frac{3}{2}\sigma_2 - \frac{\delta\sigma}{2}}{\frac{3}{2}\sigma_2 - \delta\sigma} = \frac{-1 - \frac{\delta\sigma}{3\sigma_2}}{1 - \frac{2\delta\sigma}{3\sigma_2}} \approx \left(-1 - \frac{\delta\sigma}{3\sigma_2}\right)\left(1 + \frac{2\delta\sigma}{3\sigma_2}\right) = -1 - \frac{\delta\sigma}{\sigma_2} \to -1 \quad \text{as } \delta\sigma \to 0$$

and

$$\frac{\delta\varepsilon_2}{\delta\varepsilon_1} = \frac{\frac{1}{2}\delta\sigma}{\frac{3}{2}\sigma_2 - \delta\sigma} = \frac{\frac{\delta\sigma}{3\sigma_2}}{1 - \frac{2\delta\sigma}{3\sigma_2}} \approx \frac{\delta\sigma}{3\sigma_2}\left(1 + \frac{2\delta\sigma}{3\sigma_2}\right) \to 0 \quad \text{as } \delta\sigma \to 0$$

Hence the Lévy-Mises equations show that

$$\delta\varepsilon_3 = -\delta\varepsilon_1; \qquad \delta\varepsilon_2 = 0$$

Therefore, we find that the tube does not change its length once plastic behaviour has begun. Hence we have a condition of **plane strain** during plastic behaviour – strain is restricted to the 1-3 plane.

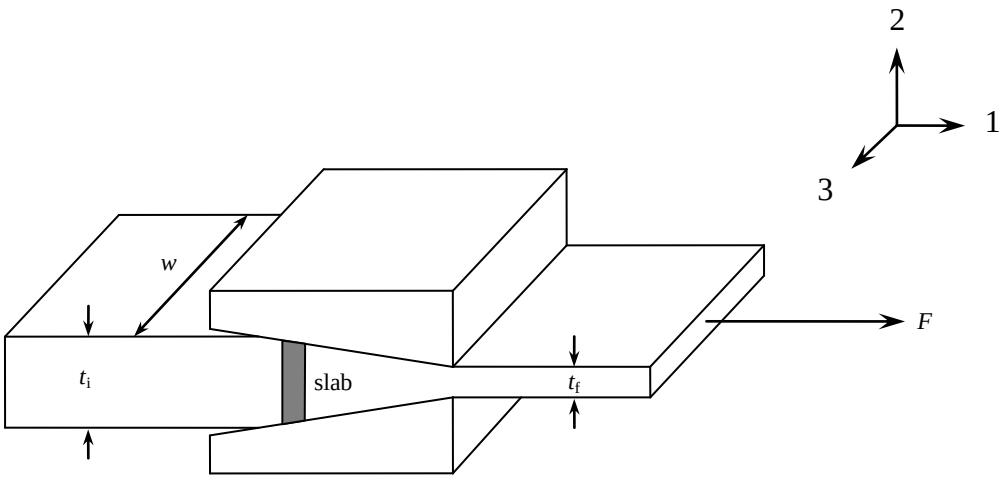
Thus, in words, plastic deformation causes the circumference of the pressure vessel to expand in length, while the tube wall thins.

Note also that the maximum shear stress is in the 1-3 plane – this plastic deformation behaviour is in accord with this.

The pressured thin-walled tube is one example of plastic behaviour, where there is **both** plane stress **and** plane strain. Other practical engineering examples arise which are examples of *either* plane stress *or* plane strain.

In **plane strain**, one principal strain (say ε_3) is zero, so that $\delta\varepsilon_3 = 0$. Such a situation arises in forging and rolling, where flow in a particular direction is constrained by other material or by a well-lubricated wall.

Thus, in rolling, all the deformation is perpendicular to the roll axes. The invariance in length of an internally pressurised tube is a second example of plane strain conditions.



Example of plane strain sheet drawing. The width *w* in the '3' direction is unchanged as a result of the drawing operation; all deformation occurs in the 1-2 plane, so that a slab shown is deformed plastically by being compressed in the '2' direction and extended in the '1' direction by the applied force, *F*.

In **plane stress**, one principal stress is zero, e.g., a material in the form of a thin sheet is subjected to uniaxial or biaxial tension. This is important in sheet metal forming. The internally pressurised thin walled cylinder is a second example of plane stress.

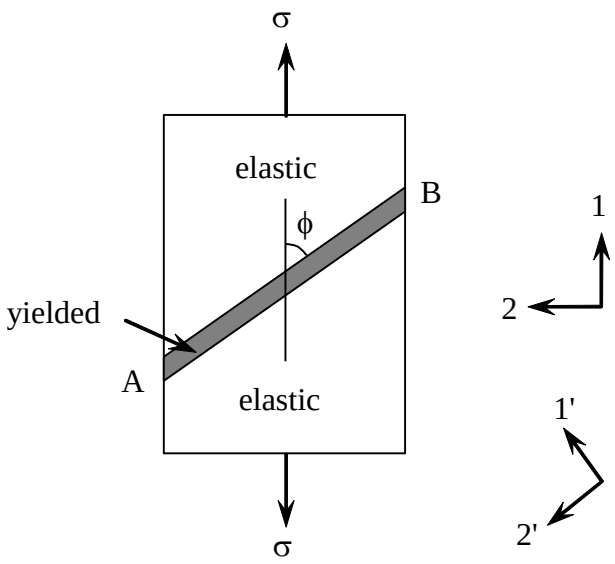
Examples of the practical deformation processes of rolling, forging, extrusion, drawing, stamping, deep drawing and pressing can be seen in the TLP 'Introduction to Deformation Processes' at

http://www.msm.cam.ac.uk/doitpoms/tlplib/metal-forming-2/index.php

Plastic deformation in plane stress

Let $\sigma_3 = 0$, i.e. consider '3' to be the direction perpendicular to the plane of a thin sheet.

If we now consider uniaxial tensile behaviour of a thin sheet, we have:



$$\sigma_1 \neq 0, \sigma_2 = \sigma_3 = 0.$$

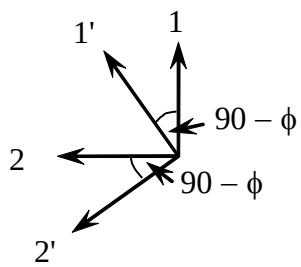
Plastic flow will start at some point within the sheet – it will not occur simultaneously all over the sheet.

Because of the constraint of neighbouring elastic material, the plastically deforming material forms in a band across the sheet at a characteristic angle ϕ to the angle of loading.

At the boundary between the elastic material and the plastically deformed material, the longitudinal strains must match for continuity. Since the strains in the elastic material are in effect zero (i.e., treating the material as a *rigid - perfectly plastic* material), the plastic strain increments along the line AB in the above diagram must also be zero, i.e.,

$$\delta\varepsilon_{AB} = 0$$

The line AB is parallel to the axis 2' which is rotated anticlockwise by $(90° - \phi)$ with respect to axis 2. Axis 1' is also rotated anticlockwise by $(90° - \phi)$ with respect to axis 1:



From the Lévy-Mises equations we have

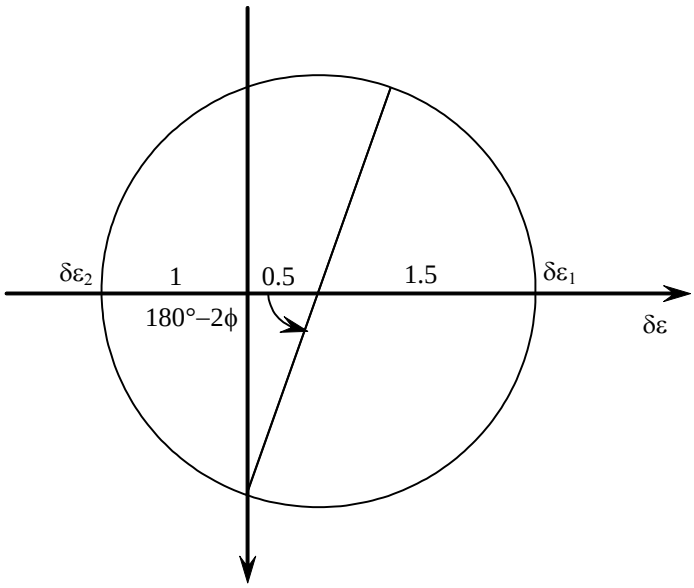
$$\frac{\delta\varepsilon_1}{\sigma_1} = \frac{\delta\varepsilon_2}{-\frac{1}{2}\sigma_1} = \frac{\delta\varepsilon_3}{-\frac{1}{2}\sigma_1}$$

and so within the plane of the sheet, we have

$$\delta\varepsilon_1 = -2\delta\varepsilon_2$$

relating incrementally the small changes $\delta\varepsilon_1$ to the incrementally small changes $\delta\varepsilon_2$ during plastic deformation.

We can represent this relationship on Mohr's circle on which $\delta\varepsilon_2 = -1$ unit and $\delta\varepsilon_1 = +2$ units of incremental strain:



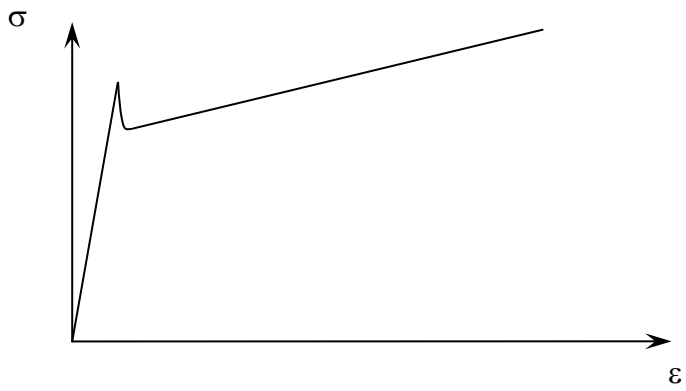
An **anticlockwise** rotation of $(180° - 2\phi)$ on the above diagram corresponds to an **anticlockwise** rotation of $(90° - \phi)$ as shown on the diagram of the thin sheet.

The intersection of the two axes in this figure then defines the direction AB for which the tensile strain $\delta\varepsilon = 0$.

From the diagram, the radius of the Mohr's circle = $1.5|\delta\varepsilon_2|$. Hence $\cos 2\phi = -1/3$, and so $2\phi = 109.47°$ and $\phi = 54.74°$.

[Note that this analysis is the same as that used by Hosford and Caddell, Third edition, p. 237, in their analysis of localised necking.]

This phenomenon of plane deformation in plane stress is well-known in mild steel:



The bands created just after the material has yielded are known as **Lüders bands**. These bands require less stress for their propagation than for their formation because of the freeing of dislocations from their solute atmospheres (c.f., Part IB).

Lüders bands occur in certain types of steel, such as low carbon steel (mild steel), but not in other metallic alloys, such as aluminium alloys and titanium alloys. This is because plastic strain localisation is normally suppressed by work hardening, which tends to make plastic flow occur rather uniformly in a metal, particularly in the early stages of plastic flow, i.e., just after yield has taken place.

However, in certain types of low carbon steel at room temperature, Cottrell atmospheres of carbon atoms which have been able to segregate preferentially to dislocation cores pin dislocations until the upper yield point is reached. Once the upper yield point is reached, there is a load drop, and then a sudden burst of plastic straining at a constant externally applied load, as cascades of dislocations are able escape their Cottrell atmospheres. This is rather specialised behaviour, caused by the ability of carbon atoms to diffuse relatively easily interstitially in these steels, but it is actually necessary behaviour for the formation of Lüders bands.

Conventional work hardening in metallic alloys in the early stages of plastic deformation makes any strain localisation (as demonstrated by the formation of Lüders bands) unlikely. This is also the case for pure metals, and for metals at high temperature, where large plastic strains can occur without a significant load increase once plastic deformation begins.

Therefore, Lüders bands only form if a limited burst of plastic straining is able to take place at constant load. Mild steels heated to sufficiently high temperatures (> 400 °C) and then tensile tested do not exhibit Lüders bands because the carbon atoms present in the mild steel are too mobile to pin the dislocations effectively.

[Note that Calladine, p. 27, incorrectly states that Lüders bands form at 45° to the tensile axis.]

Plastic deformation in plane strain

Here, one principal strain is zero. Let this be ε_3. Then, $\delta\varepsilon_3 = 0$.

From the Lévy-Mises equations it follows that

$$\sigma_3 = \frac{1}{2}(\sigma_1 + \sigma_2)$$

since

$$\frac{\delta\varepsilon_1}{\left(\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right)} = \frac{\delta\varepsilon_2}{\left(\sigma_2 - \frac{1}{2}(\sigma_3 + \sigma_1)\right)} = \frac{\delta\varepsilon_3}{\left(\sigma_3 - \frac{1}{2}(\sigma_1 + \sigma_2)\right)} \neq 0$$

[Mathematically, we can suppose that we can perturb the stresses so that

$$\sigma_3 = \frac{1}{2}(\sigma_1 + \sigma_2) + \delta\sigma$$

for some incrementally small additional stress $\delta\sigma$. Examining the behaviour in the above equations, it follows that $\delta\varepsilon_3 \to 0$ as $\delta\sigma \to 0$.]

Hence, in words, σ_3 is the mean of σ_1 and σ_2.

By convention when dealing with problems in plane strain we choose $\sigma_1 > \sigma_2$.

Therefore, $\sigma_1 > \sigma_3 > \sigma_2$.

Hence, the maximum shear stress is in the $\sigma_1 - \sigma_2$ plane at 45° to the σ_1 and σ_2 axes and is of magnitude

$$\frac{1}{2}(\sigma_1 - \sigma_2).$$

If we now examine the Tresca and von Mises yield criteria in plane strain, we find:

Tresca

$$\frac{1}{2}(\sigma_1 - \sigma_2) = k = \frac{Y}{2}$$

where k is the shear yield stress and Y the uniaxial yield stress.

Von Mises

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6k^2 = 2Y^2$$

Since $\sigma_3 = \frac{1}{2}(\sigma_1 + \sigma_2)$, we have:

$$(\sigma_1 - \sigma_2)^2 + \tfrac{1}{4}(\sigma_2 - \sigma_1)^2 + \tfrac{1}{4}(\sigma_2 - \sigma_1)^2 \equiv \tfrac{3}{2}(\sigma_1 - \sigma_2)^2 = 6k^2 = 2Y^2$$

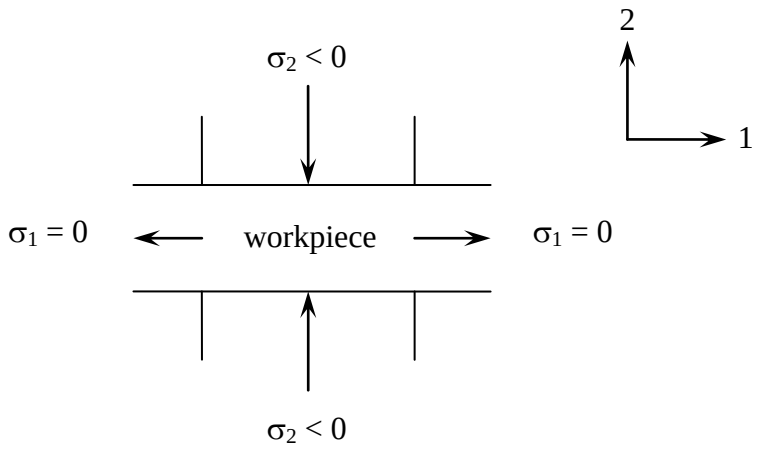
whence

$$(\sigma_1 - \sigma_2) = 2k = \frac{2Y}{\sqrt{3}}$$

Hence if we have plane strain, the Tresca yield criterion and the von Mises yield criterion have the same result expressed in *k*.

Therefore, provided we use *k*, we do not need to specify which criterion (Tresca or von Mises) we are using.

Suppose that we have uniaxial compression of a rigid – perfectly plastic material where plastic strain only takes place in the 1-2 plane, so that $\varepsilon_3 = 0$, and that there is no friction between the workpiece and the die faces (platens), e.g.:



To achieve very low friction between the workpiece and the die faces, the interface between the workpiece and the die faces must be well lubricated.

Under these circumstances,

$$\sigma_{ij} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \frac{1}{2}(\sigma_1 + \sigma_2) \end{pmatrix}$$

Therefore, the hydrostatic stress, $\sigma_{H,}$ in this situation is given by

$$\sigma_H = -p = \tfrac{1}{2}(\sigma_1 + \sigma_2) = +\sigma_3$$

where p is the hydrostatic pressure.

Therefore, at yield, we have

$$\tfrac{1}{2}(\sigma_1 - \sigma_2) = k$$

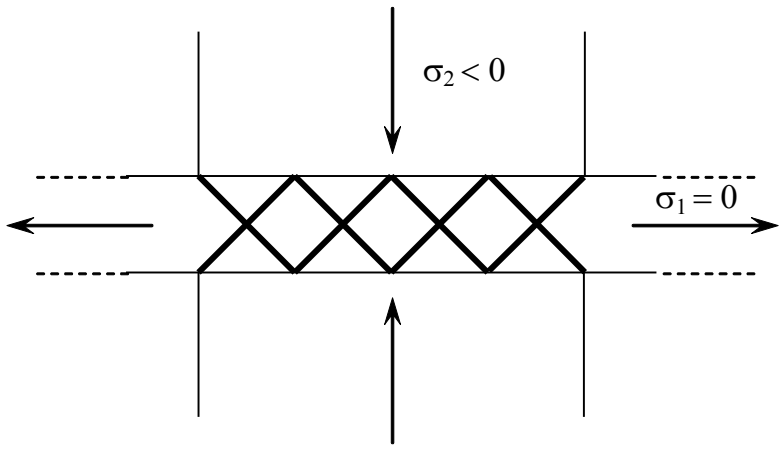
Since

$$p = -\frac{1}{2}(\sigma_1 + \sigma_2)$$

we therefore have in this particular (very special) situation that, when yielding occurs,

$\sigma_1 = -p + k = 0$
$\sigma_2 = -p - k = -2k$ since at yield $p = k$ for uniaxial compression in forging.
$\sigma_3 = -p$

The direction of maximum shear stress, here 45° to the σ_1 and σ_2 axes, are **slip lines** along which plastic sliding occurs, e.g.:



[Note: we are avoiding additional complexities here such as work hardening – remember that we have assumed the material exhibits rigid – perfectly plastic behaviour.]

The above simple example is a special case of the more general situation where the stress tensor can be written in the form

$$\sigma_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} k & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hydrostatic:	deviatioric:
(can vary in magnitude through object in general)	(pure shear yield stress: k is the same everywhere)

i.e., in the more general situation p is **not the same** as k. We will use this more general result when we examine the indentation of a material by a flat, frictionless punch.

Slip line field theory

The preceding analysis of plane strain plasticity in a simple case of uniaxial compression has just established the basis of **slip-line field theory**, which enables us to map out directions of plastic flow in plane strain plasticity problems.

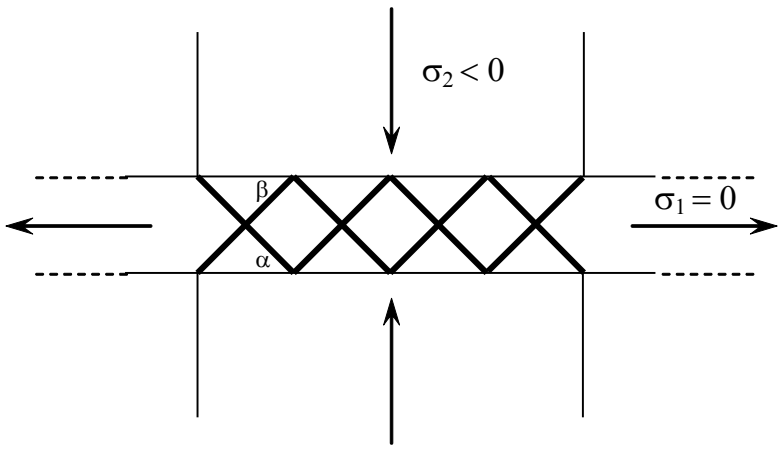
There will always be two perpendicular directions of maximum shear stress in the 1-2 plane.

These generate two families of slip lines intersecting orthogonally, as in the diagram on page 53. These are called α-lines and β-lines.

Choosing a right-handed set of orthogonal axes, so that the 1-2 plane is seen looking down the '3' axis towards the 1-2 plane, the convention for labelling the lines is as follows:

An anticlockwise rotation from α to β crosses σ_1, the maximum principal stress axis.

The example below shows the convention applied to the analysis in the simple forging situation where there is no sticking friction.



This is seen experimentally to be a realistic plastic deformation situation, e.g.:

1. PVC seen under polarised light conditions (exploiting strain birefringence)

2. Nitrogen-containing steels can be etched using Fry's reagent to reveal regions of plastic flow, e.g., in notched bars and thick-walled cylinders.

3. Under dull red heat in forging we see a distinct red cross (T > 100 °C relative to the remainder of the forging), due to dissipation of mechanical energy on the slip planes.

To develop slip-line field theory to more general plane strain conditions, we recognise that the stress can vary from point to point within the workpiece

$$\Rightarrow p \text{ can vary, but } k \text{ is a material constant}$$

Therefore, the directions of maximum shear stress and the directions of the principal stresses σ_1 and σ_2 can vary along a slip line.

The relevant equations to describe this are known as the **Hencky relations**:

The hydrostatic pressure p varies linearly with the angle ϕ turned by a slip line.

$p + 2k\phi$ is constant along an α line
$p - 2k\phi$ is constant along a β line

where the angle ϕ is in radians.

To see where these come from, we have to consider the equations of equilibrium in plane strain with respect to some fixed *x* and *y* axes in this plane.

Equations of equilibrium for plane strain:

In plane strain plasticity, we have in general tensile stresses σ_{xx} and σ_{yy} and shear stresses $\tau_{xy} = \tau_{yx}$. The shear stresses τ_{zx} and τ_{yz} are zero. The tensile stress $\sigma_{zz} = {}^1/_2(\sigma_{xx} + \sigma_{yy})$. Hence the stress tensor can be written in the form

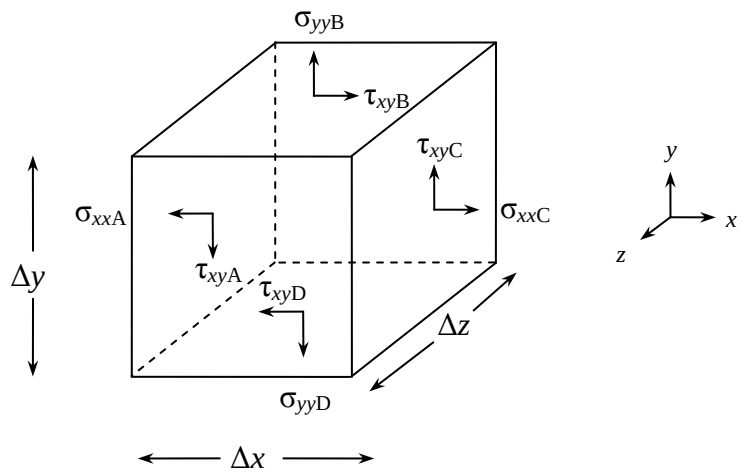
$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) \end{pmatrix}$$

It follows that the plastic strain increment $\delta\varepsilon_{zz} = 0$.

If the stress can vary from point to point, we need σ_{xx}, σ_{yy} and τ_{xy} to satisfy the equilibrium equations

$$\boxed{\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \end{aligned}}$$

To see why, consider a small element of material upon which the stress system is acting with orthogonal sides of lengths Δx, Δy and Δz, as in the diagram below. For simplicity the diagram only shows σ_{xx}, σ_{yy} and τ_{xy}, acting on faces A, B, C and D of the element; σ_{zz} is not shown.



The tensile stresses σ_{xx} on opposite faces A and C are similar, but not the same, because the stress is able to vary from point to point. The same principle applies to σ_{yy} on faces B and D and τ_{xy} on faces A, B, C and D.

Since the element is in equilibrium, the net forces in the *x*-, *y*- and *z*-directions must be zero.

If we look at the forces in the *x*-direction, it follows that

$$\left(\sigma_{xx\mathrm{C}} - \sigma_{xx\mathrm{A}}\right)\Delta y\Delta z + \left(\tau_{xy\mathrm{B}} - \tau_{xy\mathrm{D}}\right)\Delta x\Delta z = 0$$

Dividing though by $\Delta x\Delta y\Delta z$, this condition becomes

$$\frac{\left(\sigma_{xx\mathrm{C}} - \sigma_{xx\mathrm{A}}\right)}{\Delta x} + \frac{\left(\tau_{xy\mathrm{B}} - \tau_{xy\mathrm{D}}\right)}{\Delta y} = 0 \qquad (1)$$

If we define stresses σ_{xx}, σ_{yy} and τ_{xy} acting at the centre of the element, the stresses on the faces A, B, C and D can be determined by Taylor series expansions for suitably small Δx, Δy and Δz. Thus, for example,

$$\sigma_{xx\mathrm{C}} = \sigma_{xx} + \frac{1}{2}\frac{\partial\sigma_{xx}}{\partial x}\Delta x + O\left(\Delta x\right)^2$$

$$\sigma_{xx\mathrm{A}} = \sigma_{xx} - \frac{1}{2}\frac{\partial\sigma_{xx}}{\partial x}\Delta x + O\left(\Delta x\right)^2$$

$$\tau_{xy\mathrm{B}} = \tau_{xy} + \frac{1}{2}\frac{\partial\tau_{xy}}{\partial y}\Delta y + O\left(\Delta y\right)^2$$

$$\tau_{xy\mathrm{D}} = \tau_{xy} - \frac{1}{2}\frac{\partial\tau_{xy}}{\partial y}\Delta y + O\left(\Delta y\right)^2$$

Substituting these equations into (1) and letting Δx and Δy tend towards zero, we find that equation (1) reduces to the equation

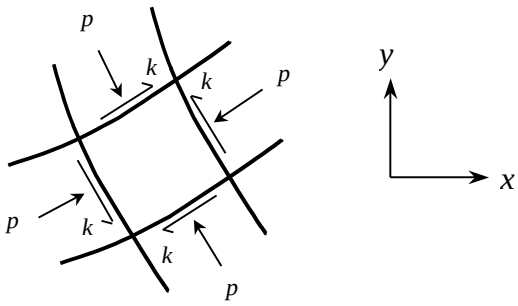
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

A similar consideration of the forces in the *y*-direction leads to the second equilibrium equation

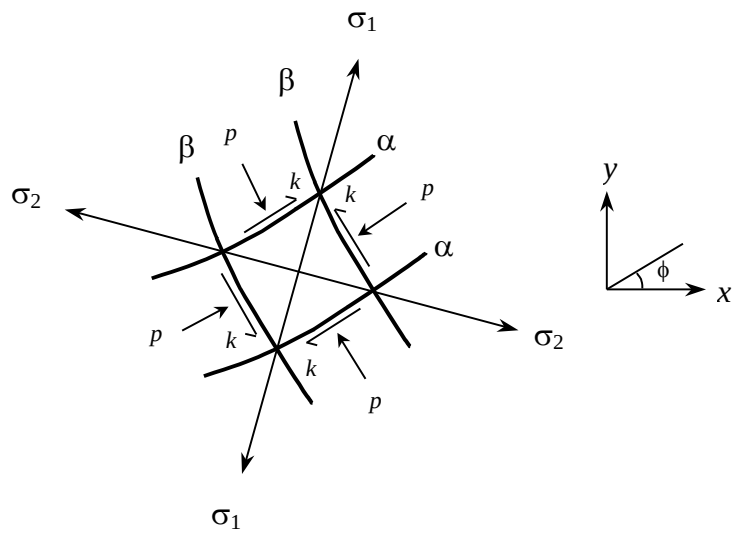
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Unless $\tau_{xy} = \pm k$, where k is the shear yield stress, the *x*- and *y*-directions (or axes) will not correspond to the directions of the α- and β-slip lines, which are themselves at ± 45° to the directions of principal stresses acting on the element.

In general, we need to examine the stresses on a small curvilinear element in the *x*-*y* plane upon which a shear stress and a hydrostatic stress are acting, and where the principal stresses are $-p-k$ and $-p+k$ for a situation where there is plane strain compression, such as in forging or indentation in which k is a constant but p can vary from point to point:



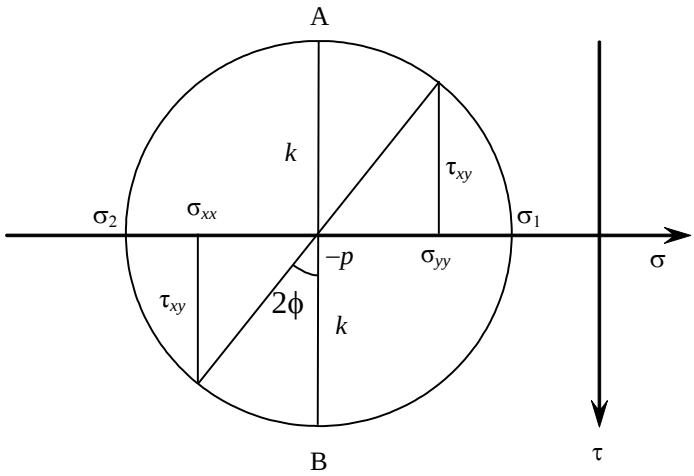
We can then identify on this diagram the directions of principal stress σ_1 and σ_2 (remembering that $\sigma_1 > \sigma_2$), and which of the lines are α-lines and which are β-lines. We can also specify the angle ϕ of the α-lines with respect to the *x*-axis:



Suppose that the α-slip line passing through the element makes an angle ϕ with respect to reference *x*- and *y*-axes, as in the above diagram. The β-slip line must then make an angle of $90° + \phi$ with respect to the *x*-axis, so that an anticlockwise rotation from the α-slip line to the β-slip line crosses the direction of maximum principal stress, σ_1.

The direction parallel to the principal stress σ_1 makes an angle of $45° + \phi$ with respect to the *x*-axis and the direction parallel to the principal stress σ_2 makes an angle of $135° + \phi \equiv 45° - \phi$ with respect to the *x*-axis.

On a Mohr's circle, this all looks like:



where A and B represent the stress states along the α- and β-lines respectively.

Hence, from the above,

$$\sigma_{xx} = -p - k\sin 2\phi$$
$$\sigma_{yy} = -p + k\sin 2\phi$$
$$\tau_{xy} = k\cos 2\phi$$

while the tensile stress in the *z*-direction in plain strain plastic yielding is simply $^1/_2(\sigma_{xx} + \sigma_{yy}) = -p$.

Substituting these expressions into the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

and recognising that k is a constant independent of x and y, we obtain two equations for p and ϕ as a function of x and y:

$$-\frac{\partial p}{\partial x} - 2k\cos 2\phi\frac{\partial \phi}{\partial x} - 2k\sin 2\phi\frac{\partial \phi}{\partial y} = 0$$
$$-2k\sin 2\phi\frac{\partial \phi}{\partial x} - \frac{\partial p}{\partial y} + 2k\cos 2\phi\frac{\partial \phi}{\partial y} = 0$$

For $\phi = 0°, 2\pi, 4\pi$, etc., in which case the α and β lines coincide with the external x- and y-axes respectively at a particular position, these equations become

$$(1) \quad \frac{\partial}{\partial x}\left(p + 2k\phi\right) = 0$$

$$(2) \quad \frac{\partial}{\partial y}\left(p - 2k\phi\right) = 0$$

Integrating these equations we find

$$(1) \quad p + 2k\phi = f_1(y) + C_1$$

$$(2) \quad p - 2k\phi = f_2(x) + C_2$$

as the most general form of the solutions of these two partial differential equations. However, we know that when ϕ is exactly zero, p must have the same value in both (1) and (2). Hence it follows that $f_1(x) = f_2(y) = 0$.

In general for points in a slip-line field we have therefore proved that the Hencky relations have to be satisfied:

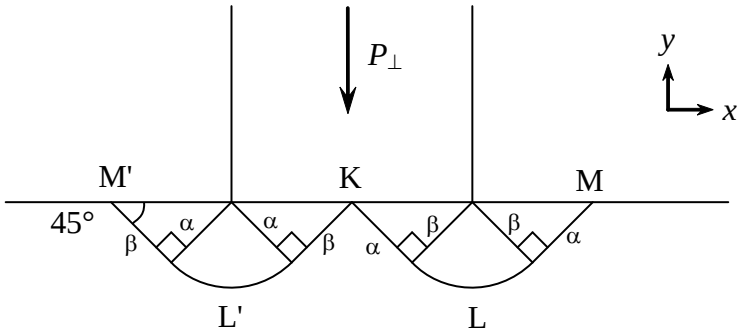
Hencky relations

The hydrostatic pressure p varies linearly with the angle ϕ turned by a slip line.

$p + 2k\phi$ is constant along an α line
$p - 2k\phi$ is constant along a β line

where the angle ϕ is in radians.

Application of the Hencky relations: indentation of a material by a flat, frictionless, punch:



In the above diagram there are free surfaces at M and M'. At both of these positions yielding will have just occurred.

There is no stress perpendicular to the free surfaces at M and M'. It follows that the slip lines must make angles of 45° to the free surfaces here and that the single (uniaxial) stress at both M and M' must be parallel to the surface.

At K the direction of maximum compressive stress, σ_2, is parallel to the *y*-axis. At M and M' the direction of maximum compressive stress is parallel to the *x*-axis. Thus, the angle turned through in radians between K and M is $^{\pi}/_2$; this has to be the same as the angle turned through in radians between K and M'.

From the definitions of the α- and β-slip lines, the curve KLM must be an α-slip line. It also follows that the curve KL'M' must be a β-slip line.

At M and M', $\sigma_2 = -2k$ because the material has just yielded (with $\sigma_1 = 0$). Hence at M, the local hydrostatic pressure, p_M, $= -^1/_2(\sigma_1 + \sigma_2) = -^1/_2(0 + (-2k)) = k$.

Using the Hencky relations,

$$p_K + 2k\phi_K = p_M + 2k\phi_M$$

where p_K is the local hydrostatic pressure at K and ϕ_K and ϕ_M are the local angles of the α-slip line with respect to the reference *x*- and *y*-axes.

Hence,

$$p_{\mathrm{K}} = p_{\mathrm{M}} + 2k\left(\phi_{\mathrm{M}} - \phi_{\mathrm{K}}\right) = k + 2k\left(\phi_{\mathrm{M}} - \phi_{\mathrm{K}}\right)$$

Now, $|\phi_{\mathrm{M}} - \phi_{\mathrm{K}}| = {}^{\pi}/_{2}$, and the hydrostatic pressure at K must be greater than that at M to have just caused yielding at M, so it follows that

$$p_{\mathrm{K}} = k + \pi k$$

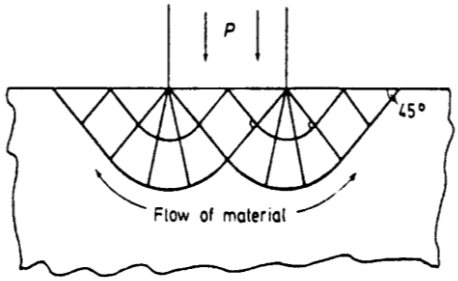
Hence at K, the maximum compressive stress (in the direction of the punch) is

$$\sigma_2 = -\,p_{\mathrm{K}} - k = -\,(2 + \pi)\,k \approx -\,5.14\,k$$

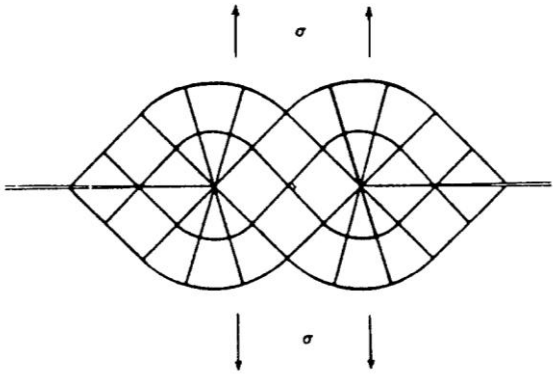
and the other (smaller in magnitude, but still compressive) principal stress at K is $\sigma_1 = -\,p_{\mathrm{K}} + k$ along the line MM'. Expressed in terms of the uniaxial yield stress, Y, of the material, the indentation pressure, $P_{\perp}$, is $2.57Y$ on the Tresca yield criterion (for which $Y = 2k$) or $2.97Y$ on the von Mises yield criterion (for which $Y = \sqrt{3}\,k$).

Examples of slip line fields

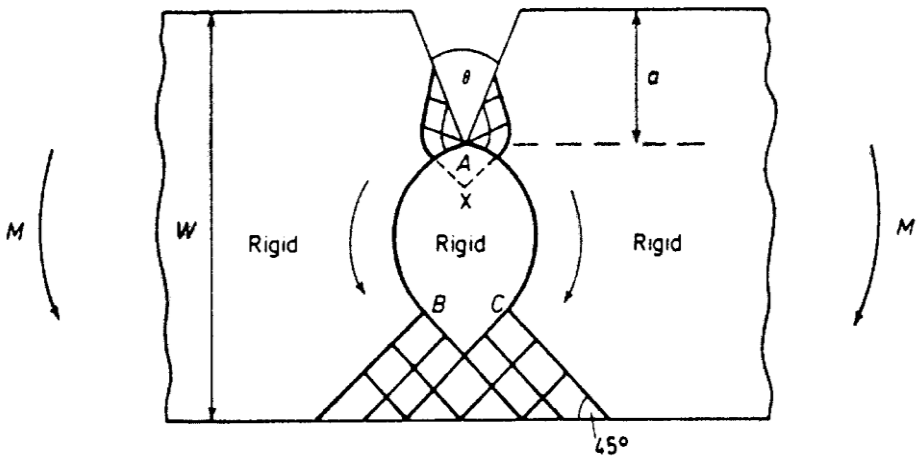
(1) Frictionless indentation with a flat punch



(2) Tension applied to a doubly slotted block of perfectly plastic material



(3) Notched bar in plane bending



Hence, in the hardness test, the pressure $P_{\perp}$, required to cause yielding beneath the punch, leading to indentation, is $\approx 3Y$.

This is because of **plastic constraint** – flow is constrained by the metal around the plastically deforming region.

Therefore, the material of the punch needs to be > 3 times stronger than the indented material if it is not to flow first.

This result is also useful in situations which are not ideal plane strain, e.g., cylindrical strain where there is axial symmetry.

Deeply notched bar

A doubly slotted block of perfectly plastic material has exactly the same slip line field as in the indentation problem. Yield occurs at $\approx 3Y$ over the notched area as in the diagram overleaf. This can lead to brittle fracture, quite apart from any elastic stress concentration effect due to the notches.

The notches raise locally the plastic flow stress, so that it may be that the raised value is above the stress σ_F required for brittle fracture.

If $\sigma_F > 3Y$, the material always flows first: the material is said to be ‘simply ductile’.

If $\sigma_F < Y$, the material always fractures first: the material is said to be ‘simply brittle’.

If $Y < \sigma_F < 3Y$, the material is ‘notch brittle’.

(using Orowan’s nomenclature).

More general plane strain situations

Other situations are more complex, e.g., a notched bar in plane bending, as in the example overleaf, but note that even here the slip lines have to meet free surfaces at 45°. This requirement is also true for **frictionless** surfaces, as in the simple example of forging.

In more general cases, the slip lines do not meet boundaries at 45° because of friction. If, in the limit, we have sticking friction (in essence a perfectly rough interface), then we get slip lines at 90° to the interface and along the interface.

Therefore, in the case of sticking friction, we have shear **along** the interface.

The slip-line patterns that we have discussed are very useful for analysing plane strain deformation in a rigid – perfectly plastic isotropic solid. However, we have not yet discussed how we arrive at the slip-line pattern.

This is the tricky bit (!). Either it is derived from model experiments in which the slip-line field is apparent or it is postulated from experience of problems with similar geometry.

For a slip-line field to be a valid solution (but not a unique solution necessarily), the stress distribution throughout the whole body, not just in the plastic region, must not violate stress equilibrium, nor must it violate the yield criterion outside the slip-line field.

The resultant velocity field must also be evaluated to ensure that strain compatibility is satisfied, i.e., matter is conserved.

These are stringent conditions and mean that obtaining a slip-line field solution is often not simple.

Instead, it is useful to take a more simple approach to the analysis of deformation processing operations where one or other of the stringent conditions is relaxed to give useful approximate solutions for part of the analysis, e.g., an estimate of the load required, or the work required.

The possible approaches are:

(1) The work formula method

(2) Limit analysis – upper bound method
– lower bound method

Work formula method

With this approach we assume that the change in geometry in a processing operation is carried out in the **most efficient way** possible.

Therefore, we estimate the **minimum** amount of work required and can also estimate the minimum force required.

Consider, for example, a uniaxial tensile deformation process of a rigid – perfectly plastic material:

In this process, $\sigma_1 = Y$, $\sigma_2 = \sigma_3 = 0$ where Y is the uniaxial yield stress.

Suppose that at some instant the bar is of length l and cross-sectional area, A. The volume $V = Al$.

If the bar is extended plastically by an amount δl, the increment of work done per unit volume, δW, is given by the expression

$$\delta W = \frac{F\delta l}{V} = \frac{F\delta l}{Al} = Y\frac{\delta l}{l} = Y\,\delta\varepsilon$$

where $\delta\varepsilon$ is the increment in true strain.

Therefore, if the bar extends from an original length l_0 to a final length l_1, the work done per unit volume, W, is

$$W = \int \delta W = Y\int_{l_0}^{l_1}\frac{\mathrm{d}l}{l} = Y\ln\frac{l_1}{l_0}$$

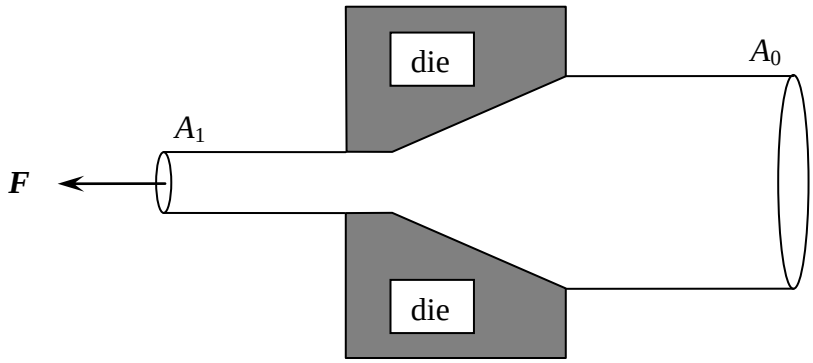
or, alternatively,

$$W = \int \delta W = Y\int_{0}^{\varepsilon_\mathrm{f}}\mathrm{d}\varepsilon = Y\varepsilon_\mathrm{f}$$

where ε_f is the final true strain.

We can apply this formula to **wire drawing**.

In wire drawing, a force is applied to the product (i.e., the wire after it has gone through the die), rather than to the billet (as in extrusion for example):



Here, the cross-sectional area of the wire is reduced from A_0 to A_1 as a result of the wire drawing process.

To extend the wire by a distance l_1, we have to feed in a length l_0 to the die, where

$$A_0 l_0 = A_1 l_1$$

by conservation of volume.

Work done by the force $F = Fl_1$

Volume of metal drawn a distance $l_1 = A_1 l_1$

From the formula for the work done/unit volume extending the bar from an original length l_0 to a final length l_1, it follows that

$$Fl_1 = (A_1 l_1) Y \ln \frac{l_1}{l_0} = Y A_1 l_1 \varepsilon_f$$

where ε_f is the final true strain attained. Hence,

$$\frac{F}{A_1} = Y \ln \frac{l_1}{l_0} = Y \ln \frac{A_0}{A_1} = Y \varepsilon_f = \sigma_{draw}$$

where σ_{draw} is the stress required – the **drawing stress**.

We can therefore estimate the maximum reduction possible with perfect lubrication. The material is rigid – perfect plastic, so $\sigma_{draw} \leq Y$ – it cannot be greater than Y.

Therefore the maximum reduction occurs when $\sigma = Y$, i.e., when

$$\ln \frac{A_0}{A_1} = \varepsilon_f = 1$$

i.e.,

$$\frac{A_0}{A_1} = e = 2.718281828... = 2.72 \text{ (3 sig. fig.)}$$

and so

$$\frac{A_1}{A_0} = 0.368 \text{ (3 sig. fig.)}$$

The fractional reduction r, is defined by the expression $r = 1 - \frac{A_1}{A_0}$.

63.2% is therefore the **maximum possible reduction** with a perfect wire-drawing process for a rigid – perfectly plastic material. Friction between the wire and the die reduces this value, causing **redundant work** – work in excess of the minimum necessary to cause the shape change.

Wire drawing of work hardening materials

If a material can work harden during the wire drawing process, so that for example, the relationship between true stress and true strain takes the form

$$\sigma = K\varepsilon^n$$

for some work hardening exponent *n*, the amount of reduction can be slightly greater than 63%. In this case the fractional reduction, *r*, is given by the formula

$$r = 1 - e^{-(1+n)}$$

To see why, we need to look first at the equation for *dW*, the work done per unit volume extending a bar by a length *dl*:

$$dW = Y\frac{dl}{l} = K\varepsilon^n d\varepsilon$$

Hence, in stretching a wire from an original length l_0 to a final length l_1 (and a final true strain ε_f), the total work done per unit volume is

$$W = \int dW = Y\frac{dl}{l} = \int_0^{\varepsilon_f} K\varepsilon^n d\varepsilon = K\frac{{\varepsilon_f}^{n+1}}{n+1}$$

where the final true strain ε_f is, as before,

$$\varepsilon_f = \ln\frac{l_1}{l_0}$$

Clearly, the situation for $n = 0$ which we have already examined for the rigid – perfect plastic material is a special case of this more general formula for *W*.

In extending the wire a distance l_1, the work done by the force = Fl_1. From the formula for the work done/unit volume extending the bar from an original length l_0 to a final length l_1, it follows that

$$Fl_1 = (A_1 l_1)\left(K\frac{{\varepsilon_f}^{n+1}}{n+1}\right)$$

Hence, the drawing stress, σ_{draw}, is now

$$\sigma_{draw} = \frac{F}{A_1} = K\frac{{\varepsilon_f}^{n+1}}{n+1}$$

and we know that $\sigma_{draw} \le$ yield stress of material at a strain of ε_f, i.e., $\sigma_{draw} \le K{\varepsilon_f}^n$.

Therefore, it follows that

$$K\frac{\varepsilon_f^{\,n+1}}{n+1} \le K\varepsilon_f{}^{n}$$

and so the fractional reduction, *r*, is now determined by the equality

$$\varepsilon_f = n+1$$

Hence

$$r = 1-\frac{A_1}{A_0} = 1-\frac{l_0}{l_1} = 1-e^{-(1+n)}$$

and so for $n > 0$, r will be greater than 0.632. Typical values of n for most metals are in the range 0.1 – 0.5. Thus, for $n = 0.1$, $r = 0.667$ and for $n = 0.5$, $r = 0.777$.

Redundant work

The theoretical drawing ratio will not be achieved in practice, even with good lubrication, In practice, the best reduction obtainable for real dies is 50% or so reduction.

To allow for redundant work, we can apply empirical corrections, e.g., by using a value of efficiency, η:

$$\eta = \frac{\text{work formula estimate}}{\text{Actual total work}}$$

and so

$$\text{Actual total work} = \frac{\text{work formula estimate}}{\eta}$$

Typical values of η are:

Extrusion: η = 45 – 55 %
Wire drawing: η = 50 – 65 %
Rolling: η = 60 – 80 %

Clearly, the work formula method gives a **lower bound** to the true force required for a given deformation processing operation because we are neglecting 'redundant work'.

For metalworking it is often preferable to have an **overestimate** of the load needed, so that we can be sure that a given operation or process is possible.

Empirical formulae allowing for redundant work give reasonably good estimates of the pressure required for processes to function.

Thus, for extrusion, the pressure required, p, is given tolerably well by the empirical equation

$$\boxed{\frac{p}{2k} = 0.8 + 1.5 \ln R}$$

where R is the ratio of areas before and after extrusion and k the shear yield stress, rather than the ideal work formula

$$\frac{p}{2k} = \ln R$$

which is obtained on the Tresca yield criterion with $Y = 2k$.

Limit analysis

This approach gives us both lower bounds and upper bounds to a deformation load required. It is much easier to apply to a problem than the slip-line field approach and is nearly as accurate.

In limit analysis we apply techniques known as ‘load bounding’ to obtain two estimates of the load needed to cause plastic flow, e.g., to cause indentation or to carry out a forging operation.

These are:

(a) The **upper-bound** value – a **definite** overestimate

(b) The **lower-bound** value – a **definite** underestimate

The true value **must** lie somewhere between these two values. If the two limits coincide, we clearly have the true value.

The upper-bound value is the one we shall look at very closely because of its attraction for analysing metalworking processes.

The lower-bound value is valuable in engineering design, e.g., when designing a structure which **must not collapse** – it provides a conservative estimate of the collapse load.

For these bounds, one of two conditions have to be satisfied:

(1) **Geometrical compatibility** between internal and external displacements or strains. This is usually concerned with kinetic conditions – we must have velocity compatibility to ensure no gain or loss of material at any point.

(2) **Stress equilibrium**: the internal stress field must balance the externally applied stresses and forces.

The basis of limit analysis rests upon the following two theorems which can be proved mathematically:

Lower-bound theorem:
If a statically admissible stress field exists such that the stresses are everywhere just below those necessary to cause yielding, then the loads associated with that field form a lower-bound solution.

Upper-bound theorem:
If a kinematically admissible strain or velocity field exists, the loads required to cause that velocity field to operate constitute an upper-bound solution.

Reasonably rigorous proofs are given by Johnson and Mellor, pp. 416–418.

In simple terms, the theorems are:

Lower bound: Any stress system which only just prevents yielding forms a lower-bound solution.

(i.e., we consider stress equilibrium, but ignore geometrical compatibility)

Upper bound: Any velocity field which can operate is associated with an upper-bound solution.

(i.e., we satisfy the geometrical compatibility, but ignore stress equilibrium)

In practice, we find an upper-bound load by postulating a deformation pattern, working out the rate at which energy is dissipated by shear in this pattern, and then equating it to the work done by the (unknown) external force.

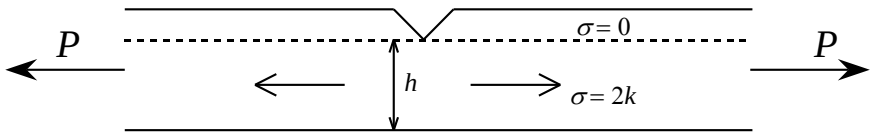
The minimum upper bound is clearly the best when considering deformation processing operations.

Examples of limit analysis

Two practical examples of the use of limit analysis are the notched bar in tension and the notched bar in plane bending:

Notched bar in tension, plane strain

Lower bound:

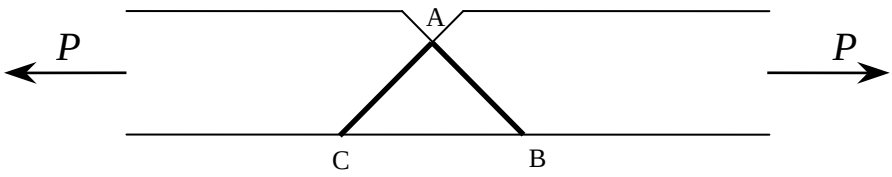


Breadth of slab = b ($b >> h$ for plane strain conditions)

Stress system assumed: $\sigma = 0$ in the length of the bar where the notch is to be found, $\sigma = 2k$ otherwise.

Equating forces (load = stress × area): $P = 2\,k\,h\,b$

Upper bound:



Assume yielding by slip on 45° shear planes AB, AC with shear stress k:

Let displacement along shear plane AB = δx

Then, internal work done = force × distance moved = $k \left|\mathrm{AB}\right| b\, \delta x = k\sqrt{2}bh\delta x$

Distance moved by the external load, P, = $\delta x \cos 45° = \delta x / \sqrt{2}$

Equating internal and external work,

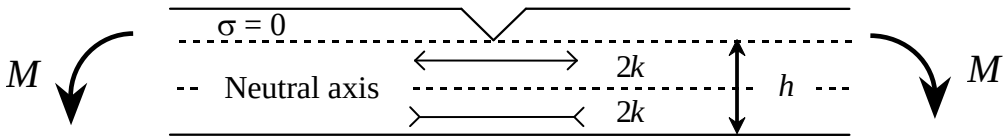
$$P\frac{\delta x}{\sqrt{2}} = k\sqrt{2}bh\delta x \text{, and so}$$

$$P = 2\,k\,h\,b$$

Hence, we obtain the same result for the upper bound and the lower bound $\Rightarrow P = 2\,k\,h\,b$ is the true failure load, i.e., the load required to cause plastic failure.

Notched bar, breadth *b*, in plane bending

Lower bound:

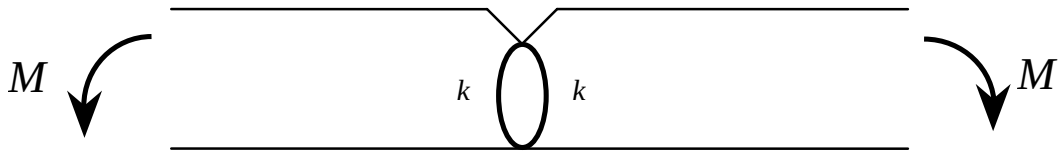


Assume stress system as above, with elements experiencing zero stress in the length of the bar where there is a notch, tensile stresses (above neutral axis) and compressive stresses (below neutral axis).

Equating couples, $$M = \left(2k.\frac{h}{2}.b\right).\frac{h}{2} = 0.5kh^2b$$

[$2k.\frac{h}{2}.b$ is the magnitude of the forces in the tensile and compressive regions, and $\frac{h}{2}$ is the distance between the centres of these two regions]

Upper bound:



Assume failure is by sliding around a 'plastic hinge' along circular arcs of length *l* and radius *r* (i.e., the blocks to the left and right of the hinge slide around the hinge).

If the rotation is $\delta\theta$, the internal work done = $k.l.b.r\delta\theta$ along one arc.

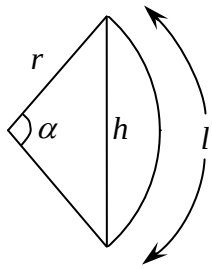
External work done = $M\delta\theta$ by one moment.

Hence,

$$M = klbr$$

but we have yet to make any assumptions about the relationship between *l* and *r*.

The upper-bound theorem tells us that **whatever** value we take for *l* and *r*, we will have an upper bound estimate. Clearly, we wish to find the lowest possible.



From the above geometry,

$$l = r\alpha \quad \text{and} \quad r = \frac{h}{2\sin(\alpha/2)}$$

Hence,

$$M = klbr = \frac{1}{4}kbh^2\,\frac{\alpha}{\sin^2(\alpha/2)}$$

so clearly for a minimum *M*, we wish to find the minimum value of the function

$$f(\alpha) = \frac{\alpha}{\sin^2(\alpha/2)}$$

Now,

$$\frac{df}{d\alpha} = \frac{1}{\sin^4(\alpha/2)}\left(\sin^2(\alpha/2) - \alpha\sin(\alpha/2)\cos(\alpha/2)\right)$$

and so this is zero when

$$\alpha = \tan(\alpha/2)$$

which can be solved numerically to give $\alpha_m = 133.57°$.

Hence, the minimum *M* is

$$M = \frac{1}{4}kbh^2\,\frac{\tan(\alpha_m/2)}{\sin^2(\alpha_m/2)} = \frac{1}{2}kbh^2\,\frac{1}{2\sin(\alpha_m/2)\cos(\alpha_m/2)} = \frac{1}{2}kbh^2\,\frac{1}{\sin\alpha_m} = 0.69kbh^2$$

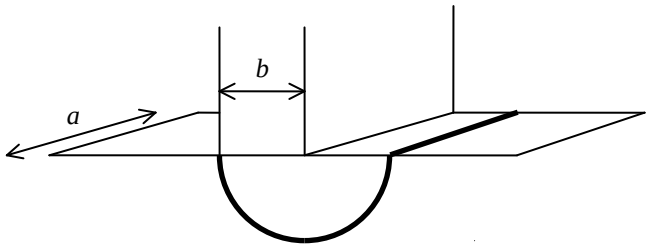
Therefore, for the notched bar in plane bending, we have the result that the moment, *M*, is such that

$$\boxed{0.5kbh^2 \le M \le 0.69kbh^2}$$

The problem of the notched bar in plane bending forms a good example of constraining the value of the external force (in this case a pure couple) between a lower bound and an upper bound. It is also a good example of how to produce a lower limit on an upper-bound calculation.

Having established the principles of how to use the upper-bound theorem, we can look at some more examples, starting with indentation. Here, we have three basic mechanisms: (a) punch not constrained, (b) punch constrained and (c) sliding over punch face:

(a) punch not constrained to be vertical, *a* >> *b*



We imagine a half-cylinder in the metal being rotated through an angle $\delta\theta$ by the punch, as shown.

External work done by external force F acting over the area ab displacing the bottom punch face by a mean amount $b\delta\theta/2$ =

$$\frac{1}{2}Fb\delta\theta$$

(the right hand side of the punch face is not displaced at all, while the left hand side will be displaced an amount $b\delta\theta$, so that the mean displacement is $b\delta\theta/2$)

Internal work shearing over a cylinder of radius b is

$k.\pi ab.b\delta\theta$ (= shear stress × area over which shear stress acts × distance moved)

Equating the internal and external work, we have:

$$\frac{1}{2}Fb\delta\theta = k\pi ab^2\delta\theta$$

Hence,

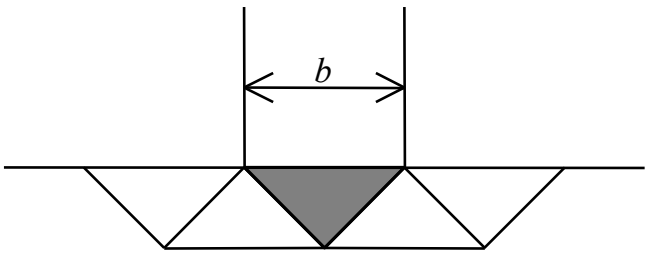
$$\frac{F}{ab} = p = 2k\pi$$

where p is the indentation pressure. This result can be written in terms of $p/2k$, i.e.,

$$\frac{p}{2k} = \pi$$

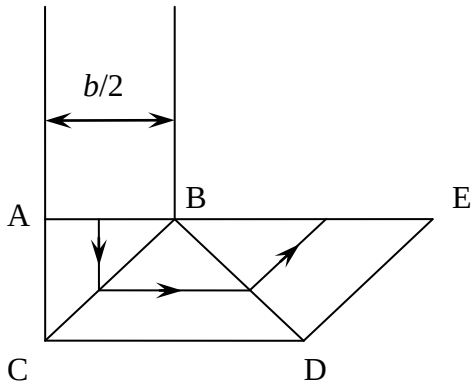
This compares with the slip-line theory result of $p/2k = 1+(\pi/2)$. The deformation pattern is very different from the slip line field. In general, an upper bound answer is better the closer the presumed deformation pattern is to the slip line field.

(b) punch constrained to be vertical – upper-bound solution



We can postulate a deformation pattern as shown above, in which we imagine five rigid blocks of metal moving in response to the applied force. The shaded block in the centre is presumed to be a 'dead metal' zone moving at the same downwards velocity as the punch. Internal energy dissipation takes place by shearing on eight shear planes.

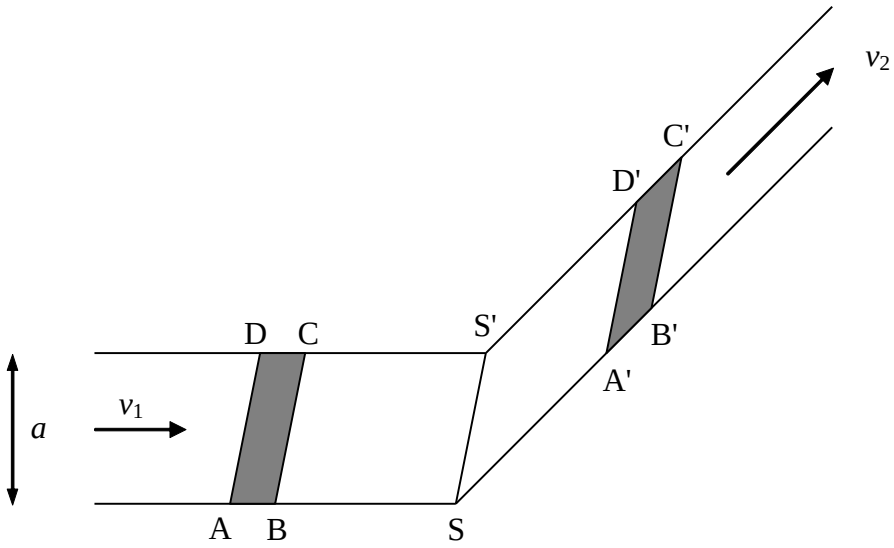
Fortunately, there is a plane of symmetry, so we need only consider one half of the problem to analyse the physical situation:



An (imaginary) particle will travel a path as shown: it is first sheared when it meets the line BC to cause it to move by shear parallel to CD. It is then sheared along DB to cause it to move by shear parallel to DE.

To analyse this, it is worth looking at the basic equation for the rate of energy dissipation in an upper bound solution in more detail.

Basic equation for energy dissipation in an upper bound solution



In the above diagram, ABCD is distorted into A'B'C'D' by shear along the vector $\overrightarrow{SS'}$ at a velocity v_s in the metal. The shear stress required to cause this deformation process is k.

Suppose ABCD moves towards the shear plane at a velocity v_1 and that there is a pressure p acting on the area al where l is the dimension out of the plane of the paper helping to cause this movement. We can take l to be unit length, but to ensure units of various quantities are correct, we will not do so.

In the above diagram, the rate of performance of work externally = $pal|v_1|$

Rate of performance of work internally = $k|\mathrm{SS'}|l|v_s|$

because the only internal work assumed is work required to effect the shear deformation so that ABCD is distorted into A'B'C'D'.

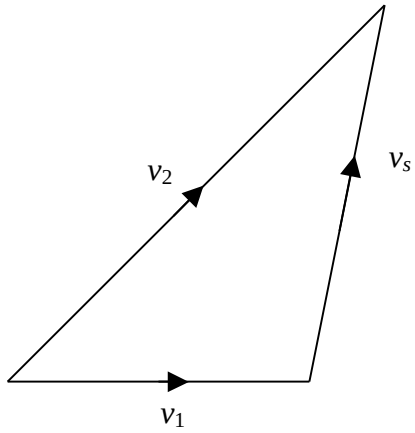
Hence equating these two quantities,

$$pal|v_1| = k|\mathrm{SS'}|l|v_s|$$

and so if we set the magnitude of the initial velocity, v_1, as unity, we find that

$$pa = k|\mathrm{SS'}||v_s|$$

Simple vector algebra relates v_1, v_2, and v_s, as on the diagram below:



If, in a more general deformation process, we have n shear planes of the type SS'$_n$ on which the shear velocities are v_{sn}, we have

$$pa = k \sum_n |\mathrm{SS'}_n \| v_{sn}|$$

The diagrams showing the relative velocities of the various parts of a deformation process are called ‘hodographs’.

Rules for constructing a hodograph

1. Label all regions of the model which move relative to each other.

2. Define an origin of the hodograph, corresponding to a stationary component of the system.

3. Draw the velocity vector of the unknown force, of unit length, on the hodograph.

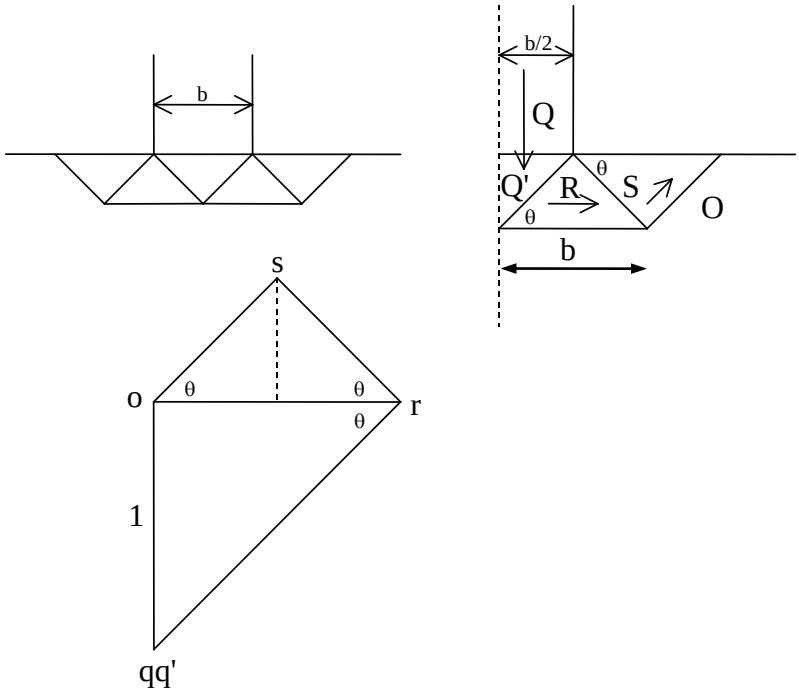
4. Draw vectors in the known directions of the moving components, relative to the origin and to each other on the hodograph.

5. Each vertex where these vectors intersect represents one (or more) of the labelled regions of the model.

6. Velocity vectors must be oriented parallel to slip planes (conservation of matter).

We are now in a position to analyse the constrained punch situation with a dead metal zone immediately below the punch face using a suitable hodograph:

(b) punch constrained – upper bound solution



The above hodograph is labelled using Bow's notation. This notation was originally developed for trusses in the 19th century by Robert Henry Bow, FRSE (1832–1908). In Bow's notation, the spaces in and around a truss are labelled, rather than the joints. The notation is introduced on page 53 of the book '*Economics of construction in relation to framed structures*' published by Bow in 1873. It turns out to be very useful for labelling hodographs. Of course, in the notation used here, regions of metal are labelled, i.e. Q, Q' R, S and O, rather than the space in and around a truss.

Punch velocity = 1.0

Q and Q' move downwards at the same unit velocity. O is a stationary component of the system, here the surrounding rigid metal.

oq and oq' are velocity vectors of unit magnitude defining the motion of particles in regions Q (the punch) and Q' (the dead metal zone).

q'r is a vector defining the shear velocity parallel to the common line of shear between Q' and R. or is the velocity of particles in region R, etc.

From the hodograph, we have for the magnitudes of the various velocities involved,

$$v_{or} = \frac{1}{\tan\theta}, \quad v_{q'r} = \frac{1}{\sin\theta}, \quad v_{os} = v_{rs} = \frac{1}{2\sin\theta}$$

Using the rate of performance of work formula for the general case, we have

$$p\frac{b}{2} = k\left\{\mathrm{Q'R}v_{\mathrm{q'r}} + \mathrm{OR}v_{\mathrm{or}} + \mathrm{RS}v_{\mathrm{rs}} + \mathrm{OS}v_{\mathrm{os}}\right\}$$

$$= k\left\{\frac{b}{2\cos\theta}.\frac{1}{\sin\theta} + b.\frac{1}{\tan\theta} + \frac{b}{2\cos\theta}.\frac{1}{2\sin\theta} + \frac{b}{2\cos\theta}.\frac{1}{2\sin\theta}\right\}$$

$$= kb\left\{\frac{1}{\sin\theta\cos\theta} + \frac{1}{\tan\theta}\right\} = kb\left\{\frac{1}{\sin\theta\cos\theta} + \frac{\cos\theta}{\sin\theta}\right\} = kb\left\{\frac{1+\cos^2\theta}{\sin\theta\cos\theta}\right\}$$

i.e.,

$$\frac{p}{2k} = \frac{1+\cos^2\theta}{\sin\theta\cos\theta}$$

This expression can be differentiated with respect to θ to find its minimum value:

$$f(\theta) = \frac{1}{\sin\theta\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\theta} = -\frac{\cos\theta}{\sin^2\theta\cos\theta} + \frac{\sin\theta}{\sin\theta\cos^2\theta} - \frac{\sin\theta}{\sin\theta} - \frac{\cos^2\theta}{\sin^2\theta} = -\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} - 1 - \frac{\cos^2\theta}{\sin^2\theta}$$

$$= \frac{1}{\cos^2\theta} - \frac{2}{\sin^2\theta}$$

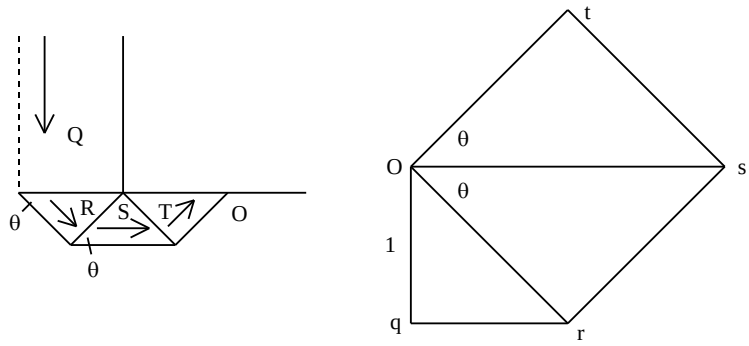
This clearly has a minimum value when $\tan\theta = \sqrt{2}$, i.e., when θ = 54.74°, for $\frac{p}{2k} = 2\sqrt{2} \approx 2.83$.

This is an improvement upon the prediction of $\frac{p}{2k} = \pi$ from the unconstrained model of punch indentation, (a).

θ/deg	30	45	54.74	60	70
$p/2k$	4.04	3.0	2.83	2.89	3.48

(c) (i) frictionless sliding over punch face

Again, the punch width is *b*, but we need only consider half of the problem:



For this situation, in the absence of friction at the punch face (so that shear with energy dissipation takes place on OR, RS, OS, ST and TO, but **not** QR), we also find:

$$\boxed{\frac{p}{2k} = \frac{1+\cos^2\theta}{\sin\theta\cos\theta}}$$

As before, oq = 1.0.

The material in R travels in the direction shown with a velocity or = 1/sinθ, etc.

From the hodograph,

$$\text{or} = \text{rs} = \text{st} = \text{ot} = \frac{1}{\sin\theta}$$

$$\text{os} = \frac{2\cos\theta}{\sin\theta} \qquad \text{qr} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

In the drawing of the indentation process,

$$\text{QR} = \text{OS} = \frac{b}{2}$$

$$\text{OR} = \text{RS} = \text{ST} = \text{OT} = \frac{b}{4\cos\theta}$$

Using the rate of performance of work formula for the general case, we have

$$p\frac{b}{2} = k\left\{\text{OR}v_{\text{or}} + \text{RS}v_{\text{rs}} + \text{OS}v_{\text{os}} + \text{ST}v_{\text{st}} + \text{OT}v_{\text{ot}}\right\}$$

$$= kb\left\{\frac{1}{4\cos\theta}.\frac{1}{\sin\theta} + \frac{1}{4\cos\theta}.\frac{1}{\sin\theta} + \frac{1}{2}.\frac{2\cos\theta}{\sin\theta} + \frac{1}{4\cos\theta}.\frac{1}{\sin\theta} + \frac{1}{4\cos\theta}.\frac{1}{\sin\theta}\right\}$$

$$= kb\left\{\frac{1}{\sin\theta\cos\theta} + \frac{\cos\theta}{\sin\theta}\right\}$$

Hence,

$$\frac{p}{2k} = \frac{1+\cos^2\theta}{\sin\theta\cos\theta}$$

as before.

(c) (ii) frictional sliding over punch face

If there is friction at the face QR, we can take it to be sticking friction, so that there is a shear stress k acting and the slippage velocity = v_{qr}.

In this case,

$$p\frac{b}{2} = k\left\{\mathrm{QR}v_{qr} + \mathrm{OR}v_{or} + \mathrm{RS}v_{rs} + \mathrm{OS}v_{os} + \mathrm{ST}v_{st} + \mathrm{OT}v_{ot}\right\}$$

$$= kb\left\{\frac{1}{2}.\frac{\cos\theta}{\sin\theta} + \left[\frac{1}{\sin\theta\cos\theta} + \frac{\cos\theta}{\sin\theta}\right]\right\}$$

$$= kb\left\{\frac{2+3\cos^2\theta}{2\sin\theta\cos\theta}\right\}$$

Hence,

$$\boxed{\frac{p}{2k} = \frac{2+3\cos^2\theta}{2\sin\theta\cos\theta}}$$

and this has a minimum value as a function of θ of $\frac{p}{2k} = \frac{7}{\sqrt{10}} + \frac{3\sqrt{2}}{2\sqrt{5}} \approx 3.16$

Therefore, of the three possible upper bound solutions we have considered where there is no frictional sliding between the indenter and the workpiece, the 'best' answer is

$$\frac{p}{2k} = 2\sqrt{2} \approx 2.83$$

which is a 10% overestimate of the slip-line field solution.

Examples of upper bound solutions for extrusion through a smooth die

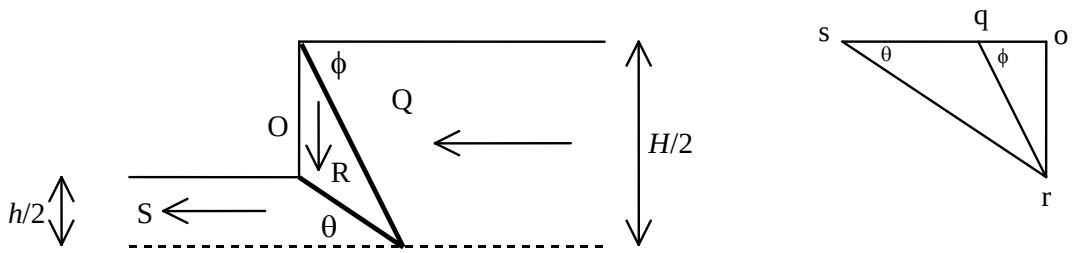
Define extrusion ratio R = ratio of areas before and after extrusion

For plane strain,

$$R = \frac{A_0}{A_1} = \frac{H}{h}$$

so that $R = 4$ corresponds to a 75% reduction in area, r.

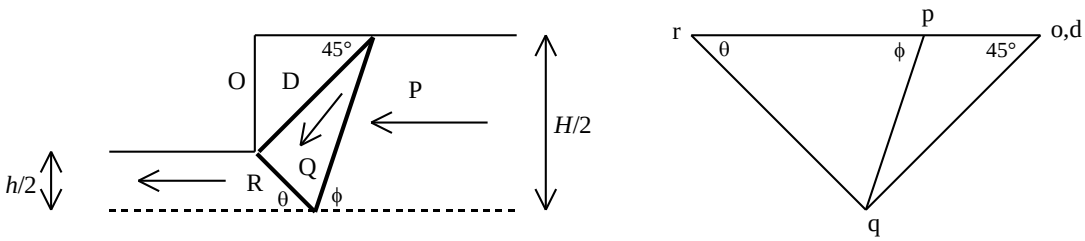
(a) square die: sliding on die face



$$p_{\min} / 2k = \frac{(R+1)}{R^{1/2}}$$

Extrusion ratio R (= H/h)	1.33	2	4	10
Reduction r	25%	50%	75%	90%
$p_{\min}/2k$	2.02	2.12	2.50	3.48
ln R (ideal work formula estimate)	0.29	0.69	1.39	2.30

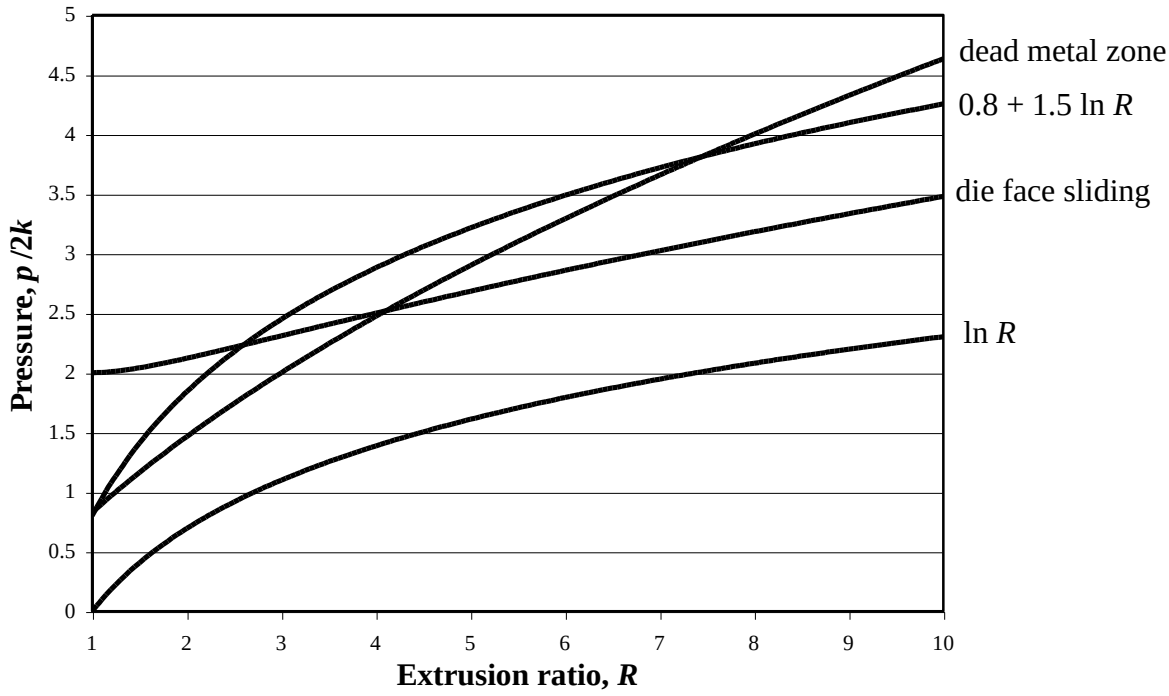
(b) square die: dead metal zone at 45°.



$$p_{\min} / 2k = 2\left[(R+1)^{1/2} - 1\right]$$

Extrusion ratio R (= H/h)	1.33	2	4	10
Reduction r	25%	50%	75%	90%
$p_{\min}/2k$	1.05	1.46	2.47	4.63

Extrusion through a smooth square die: models for extrusion pressure



The formula

$$\frac{p}{2k} = 0.8 + 1.5 \ln R$$

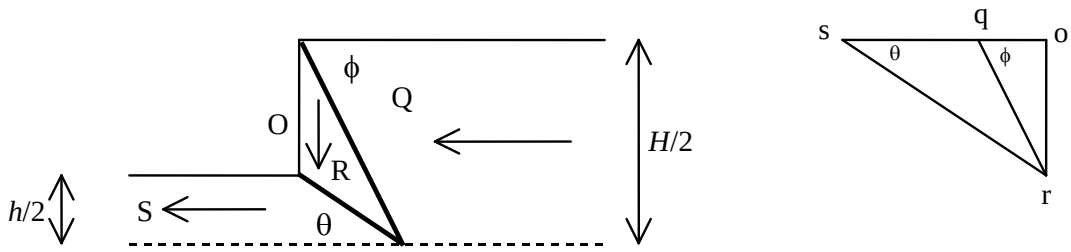
is an empirical formulae allowing for redundant work which gives reasonably good estimates of the pressure required for processes to function and is also useful for non-plane-strain problems.

The formula

$$\frac{p}{2k} = \ln R$$

is the ideal work formula estimate with the uniaxial yield stress $Y = 2k$ using the Tresca yield criterion.

Square die sliding: the mathematics behind the equation for p_{min}



To analyse this situation it is convenient to define angle osr in the hodograph as θ and the angle oqr as ϕ. Thus the plane of shear between R and S makes an angle θ with the horizontal extrusion axis, and the plane of shear between Q and R makes an angle ϕ with the horizontal extrusion axis.

Let oq be a velocity vector of unit magnitude in the hodograph, i.e., $v_{oq} = 1$. Hence,

$$v_{or} = \tan\phi, \quad v_{qr} = \frac{1}{\cos\phi}, \quad v_{rs} = \frac{\tan\phi}{\sin\theta} \quad \text{and} \quad v_{os} = R$$

where R is the extrusion ratio, = H/h.

From the above diagram of the extrusion process it follows that

$$\frac{h}{2\tan\theta} = \frac{H}{2\tan\phi}, \qquad \text{and so} \qquad R = \frac{H}{h} = \frac{\tan\phi}{\tan\theta}$$

Applying the general formula

$$pa = k\sum_{n} |\mathrm{SS'}_n \| v_{sn}|$$

we then have

$$p\frac{H}{2} = k\{\mathrm{QR}v_{qr} + \mathrm{RS}v_{rs}\}$$

where QR is the length of the line dividing regions Q and R and RS is the length of line dividing regions R and S in the diagram of the extrusion operation.

Now,

$$\mathrm{QR} = \frac{H}{2\sin\phi} \qquad \text{and} \qquad \mathrm{RS} = \frac{H}{2\sin\theta}$$

and so

$$p\frac{H}{2} = k\left\{\frac{H}{2\sin\phi}.\frac{1}{\cos\phi} + \frac{h}{2\sin\theta}.\frac{\tan\phi}{\sin\theta}\right\}$$

$$= k\left\{\frac{H}{2\sin\phi}.\frac{1}{\cos\phi}+\frac{H}{2\sin\theta}.\frac{\tan\theta}{\sin\theta}\right\}$$

substituting for $h\tan\phi$. Hence,

$$p = k\left\{\frac{1}{\sin\phi\cos\phi}+\frac{1}{\sin\theta\cos\theta}\right\}$$

Now,

$$\frac{1}{\sin\phi\cos\phi}=\frac{\sin^2\phi+\cos^2\phi}{\sin\phi\cos\phi}=\tan\phi+\frac{1}{\tan\phi}\equiv R\tan\theta+\frac{1}{R\tan\theta}$$

and likewise

$$\frac{1}{\sin\theta\cos\theta}=\tan\theta+\frac{1}{\tan\theta}$$

whence

$$p = k(R+1)\left\{\tan\theta+\frac{1}{R\tan\theta}\right\}$$

To obtain the lowest value of p, $p_{\min,}$ we minimise this function with respect to $\tan\theta$. If we let $\tan\theta = x$, then

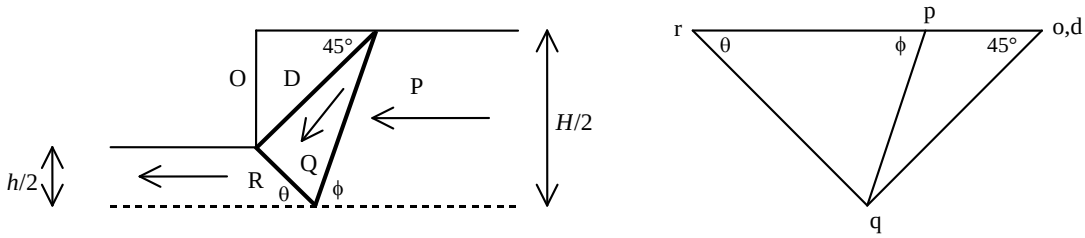
$$p = k(R+1)\left\{x+\frac{1}{Rx}\right\}$$

and so the function is a minimum when $R\,x^2 = 1$. Hence,

$$\frac{p_{\min}}{2k}=\frac{(R+1)}{\sqrt{R}}$$

for this model of extrusion in which there is sliding on the die face.

Dead metal zone at 45°: the mathematics behind the equation for p_{min}



Mathematically, this is more complicated than the situation in which there is sliding on the die face.

In the hodograph, it is convenient to define angle orq in the hodograph as θ and angle rpq as ϕ, as shown. By definition, angle poq is 45°.

Applying the general formula

$$pa = k\sum_{n}\left|\mathrm{SS'}_n\right\|v_{sn}|$$

we then have

$$p\frac{H}{2} = k\left\{\mathrm{PQ}v_{\mathrm{pq}} + \mathrm{DQ}v_{\mathrm{dq}} + \mathrm{QR}v_{\mathrm{qr}}\right\}$$

where PQ is the length of the line dividing regions P and Q, DQ is the length of line dividing regions D and Q and QR is the length of line dividing regions Q and R in the diagram above of the extrusion operation.

Now,

$$\mathrm{PQ} = \frac{H}{2\sin\phi},\ \mathrm{DQ} = \frac{1}{\sqrt{2}}\left(H-h\right)\ \text{ and }\quad \mathrm{QR} = \frac{h}{2\sin\theta}$$

In the hodograph, op = 1 and or = *R*, the extrusion ratio, i.e., $v_{\mathrm{op}} = 1$ and $v_{\mathrm{or}} = R$. From triangle opq we have, using the sine rule,

$$\frac{\mathrm{pq}}{\sin 45^\circ} = \frac{1}{\sin\left(\phi - 45^\circ\right)},\quad \text{and so}\ \ v_{\mathrm{pq}} = \frac{1}{\sin\phi - \cos\phi}$$

Likewise,

$$\frac{\mathrm{dq}}{\sin\left(180^\circ - \phi\right)} = \frac{1}{\sin\left(\phi - 45^\circ\right)},\quad \text{and so}\ \ v_{\mathrm{dq}} = \frac{\sqrt{2}\sin\phi}{\sin\phi - \cos\phi}$$

Finally, in triangle pqr, we find

$$\frac{\mathrm{pq}}{\sin\theta} = \frac{\mathrm{qr}}{\sin\phi},\quad \text{and so}\ \ v_{\mathrm{qr}} = \frac{\sin\phi}{\left(\sin\phi - \cos\phi\right)}.\frac{1}{\sin\theta}$$

Hence,

$$p\frac{H}{2} = k\left\{\frac{1}{(\sin\phi - \cos\phi)}.\frac{H}{2\sin\phi} + \frac{(H-h)\sin\phi}{(\sin\phi - \cos\phi)} + \frac{\sin\phi}{(\sin\phi - \cos\phi)}.\frac{h}{2\sin^2\theta}\right\}$$

We can eliminate θ from this expression by observing from the diagram of the extrusion operation that

$$\frac{h}{2\tan\theta} + \frac{H}{2\tan\phi} = \frac{H}{2} - \frac{h}{2}, \quad \text{i.e.,} \quad \frac{1}{\tan\theta} + \frac{R}{\tan\phi} = R - 1$$

Rearranging our expression for *p*, we have

$$pR = \frac{k}{(\sin\phi - \cos\phi)}\left\{\frac{R}{\sin\phi} + \sin\phi\left(2R - 2 + \frac{1}{\sin^2\theta}\right)\right\} = \frac{k}{(\sin\phi - \cos\phi)}\left\{\frac{R}{\sin\phi} + \sin\phi\left(2R - 1 + \cot^2\theta\right)\right\}$$

$$= \frac{k}{(\sin\phi - \cos\phi)}\left\{\frac{R}{\sin\phi} + \sin\phi\left(2R - 1 + (R - 1 - R\cot\phi)^2\right)\right\}$$

$$= \frac{kR}{(\sin\phi - \cos\phi)}\left\{\frac{R+1}{\sin\phi} - 2(R-1)\cos\phi\right\} \quad \text{after some algebra}$$

Hence,

$$\frac{p}{2k} = \frac{1}{2(\sin\phi - \cos\phi)}\left\{\frac{R+1}{\sin\phi} - 2(R-1)\cos\phi\right\}$$

and to find the minimum value of *p*, we have to minimise the right hand side with respect to ϕ. This occurs at the angle of ϕ defined by the expression

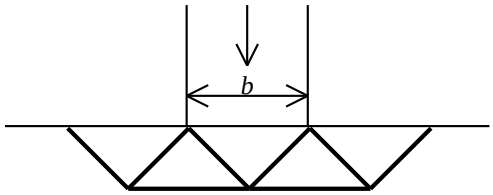
$$\cot\phi = 1 - \frac{2}{\sqrt{R+1}}$$

whence after some more algebra, we find

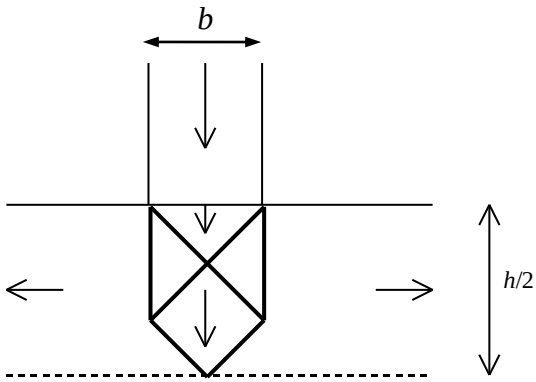
$$\frac{p_{\min}}{2k} = 2\left(\sqrt{R+1} - 1\right)$$

Examples of upper bound solutions for forging

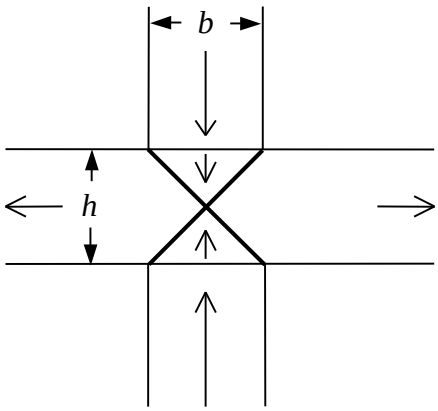
(a) *h* >> *b*



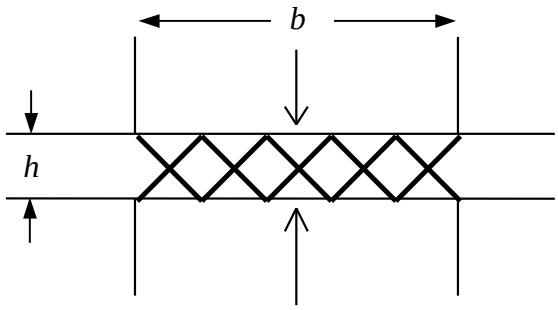
(b) *h* > *b*



(c) *h* ≈ *b*



(d) *h* << *b*



Models for forging pressure: upper bound solutions

Case (a): $p/2k = 2\sqrt{2}$

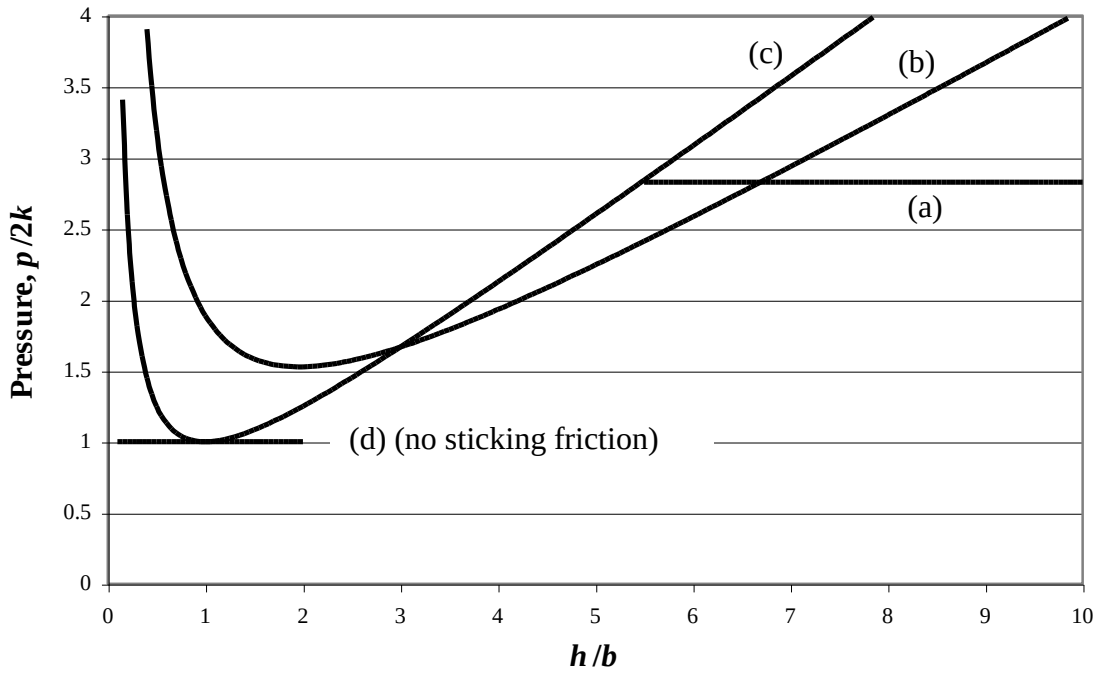
Case (b): $p/2k = \dfrac{3b}{2h} + \dfrac{7h}{18b}$

Case (c): $p/2k = \dfrac{1}{2}\left(\dfrac{b}{h} + \dfrac{h}{b}\right)$

Case (d): $p/2k = 1$ if there is no sticking friction

$= \left(1 + \dfrac{b}{4h}\right)$ if there is sticking friction (upper bound solution)

These four solutions can be plotted as shown below:

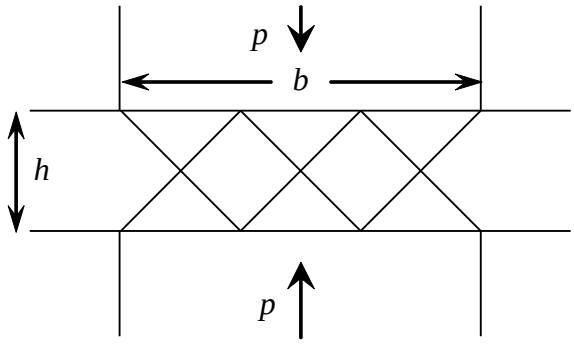


The result for (d) with sticking friction is also true if we subject the region of thickness h to tension.

Hence, if there is no void formation under hydrostatic stress and there is perfect interface adhesion, adhesive, brazed or soldered joints with plane strain geometry for $b/h \approx 100$ can have much greater tensile strengths than the bulk strength of the joining medium.

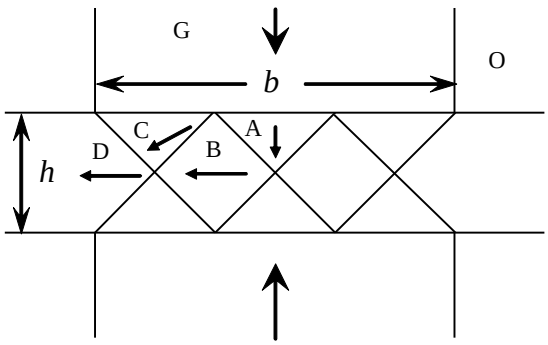
Case (d): Forging of a slab

The diagram below shows a rigid block deformation pattern for forging of a slab with $b/h = 3$.



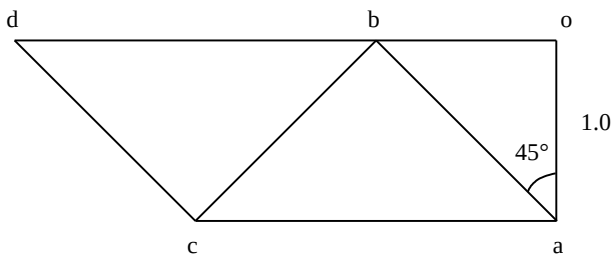
From symmetry, it is apparent that we only need to examine one quarter of the deformation pattern, such as the top left quarter. We only therefore need to draw the hodograph for this quarter.

We begin by labelling this quarter of the diagram and showing the relative movement of the various parts of the slab being forged relative to a stationary reference point O in the laboratory reference frame:



By symmetry, the central region A moves with the face of the forge, G. If there is no sticking friction, shear only occurs along planes AB, BC and CD for the top left quarter of the deformation pattern, so that the downward velocity of the faces of the forge and the central region, A, is one third of the magnitude of the velocity at which material at D in the slab moves laterally after this shearing process (since $b = 3h$).

The hodograph is straightforward:



If we take the velocity of the die faces to be unity, then in the above hodograph |oa| = 1.

From the geometry of the situation, |ab| = |bc| = |cd| = $\sqrt{2}$, |ac| = |bd| = 2 and |od| = 3.

Using the principle of upper-bound analysis, if we consider length *l* out of the plane of the paper and no friction on the die faces, we have for this top quarter:

Rate of performance of work

$$= \frac{p}{2}.l.b.\text{oa} = k.l.(\text{AB.ab} + \text{BC.bc} + \text{CD.cd})$$

since we are only considering half of the top part of the downward pressure. Hence,

$$p = \frac{2k}{b}\left(\text{AB.}\frac{\text{ab}}{\text{oa}} + \text{BC.}\frac{\text{bc}}{\text{oa}} + \text{CD.}\frac{\text{cd}}{\text{oa}}\right) = \frac{2k}{3h}\left(3\frac{h}{\sqrt{2}}.\sqrt{2}\right)$$

and so if there is no sticking friction we arrive at the result

$$\boxed{p = 2k}$$

This is also true for other values of $b/h = 2n+1$ for *n* integer provided there is no sticking friction.

Forging with sticking friction

If we have sticking friction on the die faces, we need to take account of the work done shearing material from the slab past the die faces.

For b/h = 3 for the top left hand quarter, this introduces an additional term in the formula for the rate of performance of work to take into account shear of region C past the die face, G. We now have

$$p = \frac{2k}{b}\left(\text{AB}.\frac{\text{ab}}{\text{oa}} + \text{BC}.\frac{\text{bc}}{\text{oa}} + \text{CD}.\frac{\text{cd}}{\text{oa}} + \text{GC}.\frac{\text{ac}}{\text{oa}} \right)$$

i.e.,

$$p = \frac{2k}{b}\left(3h + 2h\right) = 2k\left(1 + \frac{2}{3}\right)$$

Generalising this formula for $(b/h) = 2n+1$, for n integer, it is apparent that the formula for p in the case of sticking friction becomes

$$p = \frac{2k}{b}\left(\left(2n+1\right)h + 2h\left(1 + 2 + 3 + \ldots + n\right)\right) = 2k\left(1 + \frac{2h}{b}\sum_{i=1}^{n} i\right)$$

The summation is simply $n(n+1)/2$, and so

$$p = 2k\left(1+\frac{2hn}{b}\frac{(n+1)}{2}\right) = 2k\left(1+\frac{h}{b}n(n+1)\right)$$

Now,

$$n = \frac{b}{2h}-\frac{1}{2} \quad \text{and} \quad n+1 = \frac{b}{2h}+\frac{1}{2}$$

and so

$$\frac{p}{2k} = 1+\frac{h}{b}\left(\frac{b^2}{4h^2}-\frac{1}{4}\right) = 1+\frac{b}{4h}-\frac{h}{4b}$$

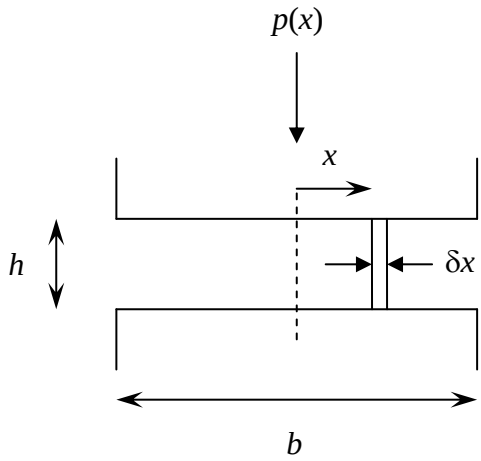
If $b/h >> 1$, the term in $h/4b$ becomes negligible, so that we arrive at the result quoted on page 97:

$$\boxed{\frac{p}{2k} = 1+\frac{b}{4h}}$$

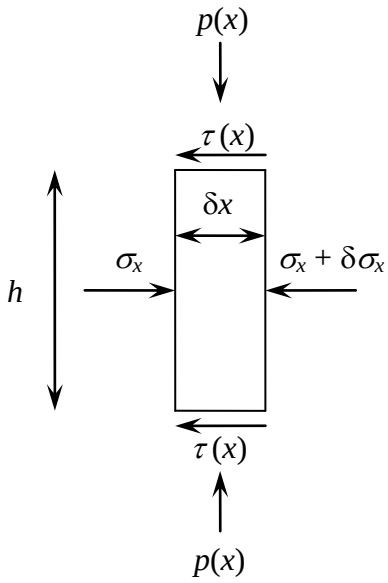
Slab analysis of forging with friction

An alternative analysis of forging to upper bound solutions via hodographs considers the diagram below. The workpiece (slab) being forged is within two parallel lubricated dies. However, friction is unavoidable between the slab and the dies, and needs to be taken into account in the analysis.

As before, we let the slab height be h and its width be b. There is a pressure distribution along the die, $p(x)$, to enable the slab to be forged. Plane strain conditions can be assumed, so that the strain perpendicular to the page of the slab is zero.



In this analysis it is convenient to focus on the magnitudes of the pressure $p(x)$ and the compressive stress $\sigma(x)$ in the slab. By symmetry we need only consider $x > 0$: the situation for $x < 0$ will be a mirror in the line $x = 0$ of the situation for $x > 0$.



If we consider the equilibrium of an element of width δx, height h and unit depth into the page at a general position, x (> 0), as in the diagram above, we have:

$$\delta\sigma_x.h.1 + 2\tau(x).\delta x.1 = 0$$

$$\Rightarrow \quad \frac{\mathrm{d}\sigma_x}{\mathrm{d}x} = -\frac{2\tau}{h}$$

i.e., the magnitude of the compressive stress σ_x decreases from the centre of the slab to the edge at $x = b/2$ where σ_x is zero.

If the shear stress τ is small in comparison with the magnitudes of p and σ_x at a position x, then, to a good approximation, the principal stresses in the x-y plane at x are p and σ_x.

Hence, in plane strain using the Tresca yield criterion, we have:

$$p - \sigma_x = Y = 2k$$

where Y is the uniaxial yield stress and k is the shear yield stress, since in plane strain the third principal stress at x coming out of the paper has then to be the mean of p and σ_x.

Differentiating this equation with respect to x, we have

$$\frac{\mathrm{d}p}{\mathrm{d}x} - \frac{\mathrm{d}\sigma_x}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}x} = 0$$

because the yield stress is position-independent for a material which is rigid–perfect plastic, i.e., one which does not work harden. We can therefore rewrite the equation

$$\frac{\mathrm{d}\sigma_x}{\mathrm{d}x} = -\frac{2\tau}{h}$$

in the form

$$\boxed{\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{2\tau}{h}}$$

We can now choose our friction law to integrate this equation. We will examine two cases, that of Coulomb friction and the situation where friction scales with the shear yield stress, k.

Coulomb friction

If we choose Coulomb friction, $\tau = \mu p$, where μ is the coefficient of friction. Hence, we have

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{2\mu}{h}\,p$$

which can be rearranged in the form

$$\frac{\mathrm{d}p}{p} = -\frac{2\mu}{h}\,\mathrm{d}x$$

Integrating we have for $x > 0$,

$$\ln p = -\frac{2\mu}{h}x + \lambda$$

where λ is a constant of integration. Clearly, for forging to occur, the slab has to (just) yield at the edges of the slab, i.e., at $x = b/2$ if $x > 0$. Therefore, at $x = b/2$, $p = Y = 2k$, and so

$$\ln 2k = -\frac{2\mu}{h}\frac{b}{2} + \lambda$$

Hence, in general, for $0 < x < b/2$,

$$\ln p = -\frac{2\mu}{h}x + \ln 2k + \frac{2\mu}{h}\frac{b}{2}$$

which can be rearranged in the form

$$\frac{p}{2k} = \exp\left(\frac{2\mu}{h}\left(\frac{b}{2} - x\right)\right)$$

For $-b/2 < x < 0$,

$$\frac{p}{2k} = \exp\left(\frac{2\mu}{h}\left(\frac{b}{2} + x\right)\right)$$

so that at $x = -b/2$, $p = 2k$. We therefore have the result that the pressure varies exponentially across the slab – there is a ‘friction hill’ associated with friction between the dies and the slab.

Friction scaling with the shear yield stress

Under these circumstances, $\tau = mk$ where $0 < m < 1$. If $m = 1$ we have sticking friction. Hence,

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{2mk}{h}$$

and so

$$p = -\frac{2mk}{h}x + \lambda$$

where λ is a constant of integration. Since $p = 2k$ at $x = b/2$,

$$2k = -\frac{2mk}{h}\frac{b}{2} + \lambda$$

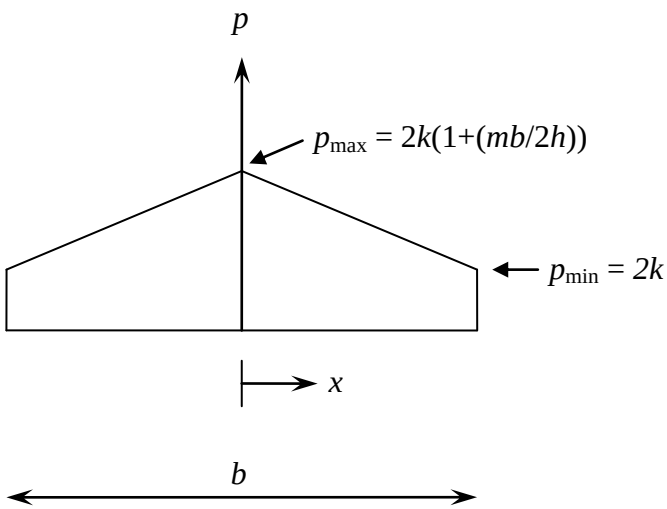
and so, in general, for $0 < x < b/2$,

$$\boxed{\frac{p}{2k} = 1 + \frac{m}{h}\left(\frac{b}{2} - x\right)}$$

For $-b/2 < x < 0$, we have by symmetry,

$$\frac{p}{2k} = 1 + \frac{m}{h}\left(\frac{b}{2} + x\right)$$

and so in this case we have a linear ‘friction hill’ describing the pressure as a function of x across the slab:



The mean pressure, p_{AV}, for $0 < x < b/2$ is then

$$p_{AV} = \frac{2}{b}\int_0^{b/2} p(x)dx$$

Since p is a linear function of x, we have, by inspection for $0 < x < b/2$, the result

$$\frac{p_{AV}}{2k} = 1 + \frac{mb}{4h}$$

and because of the mirror symmetry in the physical situation, this average pressure is valid for the domain $-b/2 < x < b/2$.

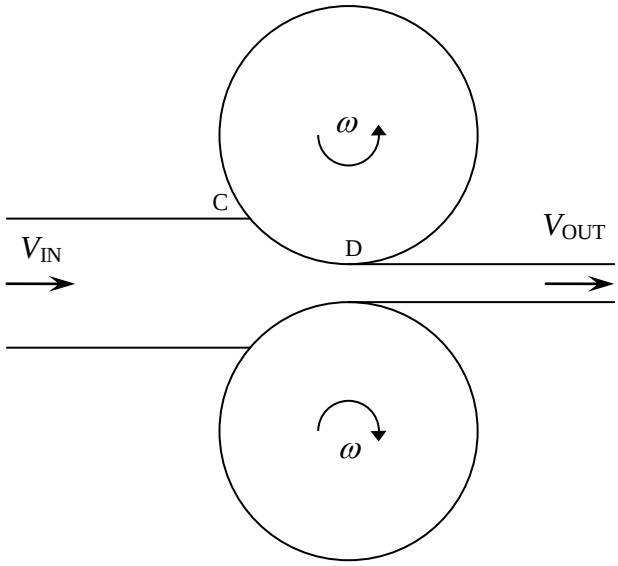
If $m = 1$ (i.e., sticking friction), we find

$$\boxed{\frac{p_{AV}}{2k} = 1 + \frac{b}{4h}}$$

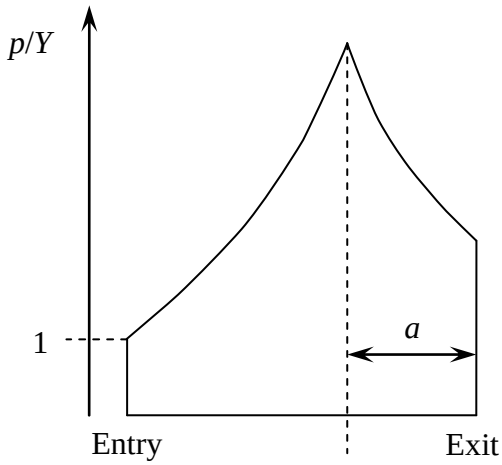
which is the same result that we have obtained from the upper bound solution via hodographs.

Rolling loads and torques: analogy with forging

In rolling there is also a ‘friction hill’. The sense of the frictional force reverses along the arc of contact, with a maximum pressure in the friction hill between the entrance point C and exit point D of the rolling mill.



At C, $V_{IN} < \omega R$, where R is the radius of each roller. At D, $V_{OUT} > \omega R$. The position where both the rollers and the strip have the same speed is the ‘no slip’ point, where the pressure p on the strip is a maximum:



Work hardening of the metal strip as it passes through the rollers means that the yield stress of the strip on exit is higher than on entrance, as in the above schematic.

Since the total rolling load, P, across the arc of contact can be considered to be concentrated at the no-slip point a distance a from the line joining the centre of the rollers, it follows that the two rollers together supply a torque $Q = 2Pa$ to enable the rolling operation to take place.

Applying a horizontal tension to the strip material being rolled can significantly reduce the rolling load – this can be appreciated quite simply in a plane strain rolling situation using the Tresca yield criterion.

Finite Element Modelling (FEM)

Finite element modelling has nowadays superseded upper-bound and slip line field methods for realistic, rather than idealised, metalforming processes.

It is a very powerful, but computationally intensive, technique. It can deal with complex geometries and enables stress distributions, temperature distributions and operational parameters to be specified.

However, to be able to use such programmes, you need to be able to understand the basics (e.g., to check your computer predictions are reasonable!), which is what I have attempted to put across in this course.

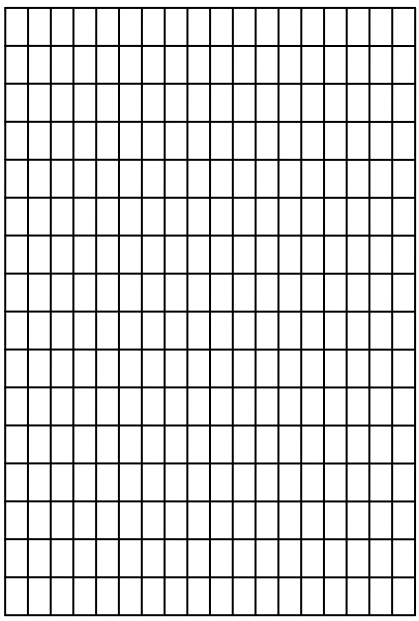
Example of FEM: The production of gudgeon pins (small lengths of tube)

Gudgeon pins are pins which hold a piston rod and a connecting rod together.

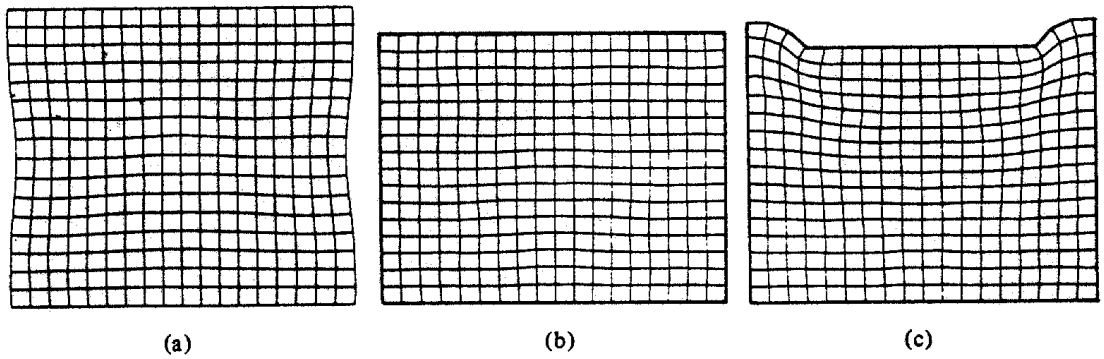
The production process consists of (i) upsetting, in which the starting billet is compressed (squashed) until it fills the die into which it has been put, (ii) indentation of the dumped billet to guide the punch in the back extrusion operation, and finally (iii) backward extrusion of the indented slug.

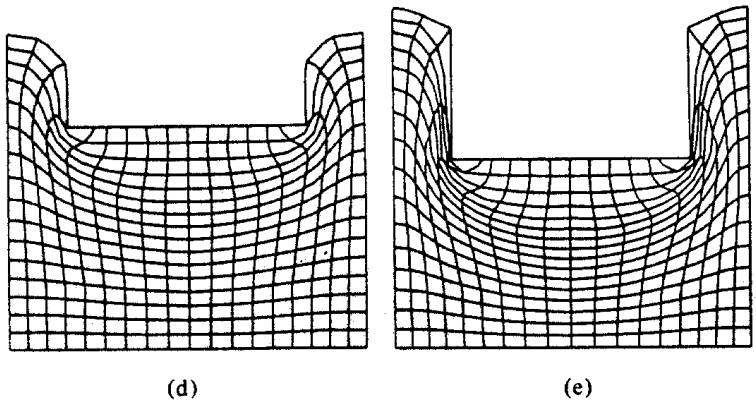
After this production process the base of the cup formed at the base of the die is punched out to produce the gudgeon pin.

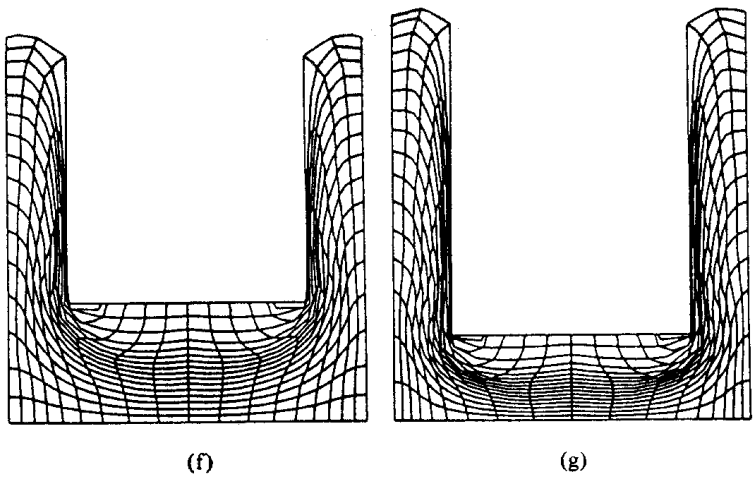
The process can be visualised very well with the help of the finite element (FE) mesh. The grid distortions during the various process stages agree very well indeed with experimentally observed distorted grids.



Initial FE mesh for the forming of a gudgeon pin







(a) – (g) Predicted FE mesh distortions for the upsetting, indentation and backward extrusion stages in the forming of a gudgeon pin.

(from G.W. Rowe, C.E.N. Sturgess, P. Hartley and I. Pillinger, *Finite-element Plasticity and Metalforming Analysis*, CUP, 1991. Reproduced with permission from Cambridge University Press.)

Appendix: Mathematics of the modified von Mises yield criterion

In principal stress space, the modified von Mises criterion takes the form

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6(k_0 - \mu\sigma_H)^2$$

where the orthogonal principal stresses are σ_1, σ_2 and σ_3, μ is a dimensionless pressure coefficient, k_0 is the shear yield stress at zero hydrostatic pressure and

$$\sigma_H = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

is a hydrostatic stress. The shape of this modified von Mises criterion is that of a circular sectional cone with its axis along $\sigma_1 = \sigma_2 = \sigma_3$.

To appreciate why this the shape of this yield surface in principal stress space, it is convenient to transform to a set of orthogonal axes **i**, **j** and **k** where **k** is along the axis of the cone and **i** and **j** are perpendicular to this., e.g.,

$$\mathbf{i} = \frac{1}{\sqrt{2}}\mathbf{i}' - \frac{1}{\sqrt{2}}\mathbf{j}'$$
$$\mathbf{j} = \frac{1}{\sqrt{6}}\mathbf{i}' + \frac{1}{\sqrt{6}}\mathbf{j}' - \frac{2}{\sqrt{6}}\mathbf{k}'$$
$$\mathbf{k} = \frac{1}{\sqrt{3}}\mathbf{i}' + \frac{1}{\sqrt{3}}\mathbf{j}' + \frac{1}{\sqrt{3}}\mathbf{k}'$$

where **i**', **j**' and **k**' are unit lengths along the σ_1, σ_2 and σ_3 axes respectively. A vector [*x*, *y*, *z*] referred to axes **i**, **j** and **k** is related to the description [σ_1, σ_2, σ_3] in the axis system **i**', **j**' and **k**' through the equation

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

(see, for example, Kelly, Groves and Kidd, *Crystallography and Crystal Defects*, Appendix 4), and conversely,

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix}$$

Hence,

$$\sigma_1 - \sigma_2 = \sqrt{2}\, x$$
$$\sigma_1 - \sigma_3 = \frac{1}{\sqrt{2}} x + \frac{3}{\sqrt{6}} y$$
$$\sigma_2 - \sigma_3 = -\frac{1}{\sqrt{2}} x + \frac{3}{\sqrt{6}} y$$

$$\sigma_1 + \sigma_2 + \sigma_3 = \sqrt{3}\, z$$

and so the modified von Mises criterion becomes

$$2x^2 + \left(\frac{x}{\sqrt{2}} + \frac{3y}{\sqrt{6}} \right)^2 + \left(-\frac{x}{\sqrt{2}} + \frac{3y}{\sqrt{6}} \right)^2 = 6\left(k_0 - \frac{\mu}{\sqrt{3}} z \right)^2$$

i.e.,

$$\boxed{x^2 + y^2 = 2\left(k_0 - \frac{\mu\, z}{\sqrt{3}} \right)^2}$$

If $\mu = 0$, the surface becomes

$$x^2 + y^2 = 2{k_0}^2$$

i.e., independent of z. Hence, as an aside, we have shown that the von Mises yield criterion is a cylinder along the z axis with a radius of $\sqrt{2}\, k_0$.

Returning to the modified von Mises criterion for $\mu \neq 0$, it is apparent from the right hand side of the equation in the above box that the radius of the cylinder is a function of z.

When $k_0 = \mu\, z / \sqrt{3}$ we have a point. Either side of this the locus of the equation is a cone (although we can ignore one of these cones as being unphysical by only considering solutions for which $k_0 \geq \mu\, z / \sqrt{3}$.

Suppose $z = z_0$ at the point. The modified von Mises yield criterion then becomes

$$x^2 + y^2 = 2{k_0}^2 \left(1 - \frac{z}{z_0} \right)^2$$

and so the semi-angle of the cone, θ, is then defined by the equation

$$\boxed{\tan\theta = \frac{\sqrt{2} k_0}{z_0} = \sqrt{\frac{2}{3}}\, \mu}$$

The modified von Mises criterion in plane stress

In plane stress (e.g., if $\sigma_3 = 0$), the modified von Mises criterion becomes

$$(\sigma_1 - \sigma_2)^2 + {\sigma_2}^2 + {\sigma_1}^2 = 6\left(k_0 - \frac{\mu}{3}(\sigma_1 + \sigma_2)\right)^2$$

i.e.,

$${\sigma_1}^2 + {\sigma_2}^2 - \sigma_1\sigma_2 = 3\left(k_0 - \frac{\mu}{3}(\sigma_1 + \sigma_2)\right)^2$$

If we transform the modified von Mises yield criterion to *X*- and *Y*-axes at 45° to the σ_1 and σ_2 axes, a coordinate position (σ_1, σ_2) will be transformed into a coordinate position (*X*, *Y*) so that

$$\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\cos 45° & -\sin 45°\\ \sin 45° & \cos 45°\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}$$

i.e.,

$$\sigma_1 = \frac{1}{\sqrt{2}}(X - Y)$$
$$\sigma_2 = \frac{1}{\sqrt{2}}(X + Y)$$

Hence substituting into the equation

$${\sigma_1}^2 + {\sigma_2}^2 - \sigma_1\sigma_2 = 3\left(k_0 - \frac{\mu}{3}(\sigma_1 + \sigma_2)\right)^2$$

we have

$$\frac{1}{2}(X - Y)^2 + \frac{1}{2}(X + Y)^2 - \frac{1}{2}(X - Y)(X + Y) = 3\left(k_0 - \frac{\mu\sqrt{2}}{3}X\right)^2$$

i.e.,

$$\frac{X^2}{2} + \frac{3Y^2}{2} = 3\left(k_0 - \frac{\mu\sqrt{2}}{3}X\right)^2$$

i.e.,

$$\frac{X^2}{2}+\frac{3Y^2}{2}=3\left(k_0{}^2+\frac{2\mu^2}{9}X^2-\frac{2\mu\sqrt{2}}{3}k_0X\right)$$

Rearranging this, we have

$$X^2\left(\frac{1}{2}-\frac{2\mu^2}{3}\right)+2\mu\sqrt{2}k_0X+\frac{3Y^2}{2}=3k_0{}^2 \qquad [1]$$

This can be manipulated to be in the form

$$\frac{(X+P)^2}{a^2}+\frac{Y^2}{b^2}=1 \qquad [2]$$

i.e., an ellipse centred at (–*P*, 0) with major and minor axes *a* and *b* respectively, as in the schematic on page 40.

[Note that when $\mu = 0$ the surface is simply

$$\frac{X^2}{6k_0{}^2}+\frac{Y^2}{2k_0{}^2}=1$$

which describes an ellipse centred at the origin (0, 0) with a major axis $\sqrt{6}\,k_0$ and a minor axis $\sqrt{2}\,k_0$, i.e., an ellipse where the ratio of the major axis to the minor axis is $\sqrt{3}$: 1]

Returning to equation [2], this is an equation of the form

$$\alpha X^2b^2+2\alpha b^2PX+\alpha a^2Y^2=\alpha b^2\left(a^2-P^2\right) \qquad [3]$$

where each term has been multiplied by a constant α, to be evaluated.

Comparing [1] and [3], it is apparent that we have four equations for the four unknowns *a*, *b*, α and *P*:

$$\alpha b^2=\left(\frac{1}{2}-\frac{2\mu^2}{3}\right)$$
$$\alpha b^2P=\mu\sqrt{2}k_0$$
$$\alpha a^2=\frac{3}{2}$$
$$\alpha b^2\left(a^2-P^2\right)=3k_0{}^2$$

Hence,

$$P = \frac{\mu\sqrt{2}k_0}{\left(\frac{1}{2} - \frac{2\mu^2}{3}\right)}$$

$$a^2 = \frac{3}{2\alpha}$$

$$b^2 = \frac{1}{\alpha}\left(\frac{1}{2} - \frac{2\mu^2}{3}\right)$$

$$\frac{3}{2\alpha}\left(\frac{1}{2} - \frac{2\mu^2}{3}\right) - \frac{2\mu^2 {k_0}^2}{\left(\frac{1}{2} - \frac{2\mu^2}{3}\right)} = 3{k_0}^2$$

Rearranging the last equation, we have:

$$\frac{3}{2\alpha}\left(\frac{1}{2} - \frac{2\mu^2}{3}\right)^2 - 2\mu^2 {k_0}^2 = 3{k_0}^2\left(\frac{1}{2} - \frac{2\mu^2}{3}\right) = \frac{3}{2}{k_0}^2 - 2\mu^2 {k_0}^2$$

whence

$$\alpha = \frac{1}{{k_0}^2}\left(\frac{1}{2} - \frac{2\mu^2}{3}\right)^2 = \frac{1}{4{k_0}^2}\left(1 - \frac{4\mu^2}{3}\right)^2$$

and so

$$a = \sqrt{\frac{3}{2\alpha}} = \frac{\sqrt{6}k_0}{\left(1 - \frac{4\mu^2}{3}\right)}$$

$$b = \frac{\sqrt{2}k_0}{\left(1 - \frac{4\mu^2}{3}\right)^{1/2}}$$

$$P = \frac{2\sqrt{2}\,\mu k_0}{\left(1 - \frac{4\mu^2}{3}\right)}$$

Question Sheet 1

1. The relative displacement tensor, e_{ij}, associated with slip on a single slip system is given by

$$e_{ij} = \gamma n_j \beta_i$$

where γ is the shear strain angle (assumed to be small), **n** is a unit vector normal to the slip plane and $\boldsymbol{\beta}$ is a unit vector in the slip direction.

Write down the components of the strain tensor, ε_{ij}, associated with the deformation.

Lead sulphide, PbS (galena), is cubic holosymmetric (i.e., it has the maximum point group symmetry associated with cubic crystals) and slips on <011>{100}, even though it has the rock salt structure. Write down strain tensors corresponding to slip on the six physically distinct slip systems. How many of these slip systems are independent? What can you deduce about the ductility of PbS?

2. The state of stress imposed on a metal is defined by the principal stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Using suitable equations from the Tensor Properties course (C4), determine the magnitude of the shear stress on the $(11\bar{1})$ plane. Show that this is the same as the magnitude of the shear stress on the (111), $(1\bar{1}1)$ and $(\bar{1}11)$ planes.

Show that yielding according to the von Mises criterion occurs when:

(i) the magnitudes of the shear stresses on each of the octahedral {111} planes reaches a critical value, C_1, and

(ii) the quadratic invariant, J_2, of the deviatoric stress tensor derived from σ_{ij} reaches a critical value, C_2.

Determine the values of C_1 and C_2 in terms of the uniaxial yield stress, Y, of the metal.

3. The yield stress of an aluminium alloy in uniaxial tension is 320 MPa. The same alloy also yields under the combined stress state

$$\sigma_{ij} = \begin{pmatrix} 330 & 95 & 0 \\ 95 & 70 & 0 \\ 0 & 0 & 150 \end{pmatrix} \text{MPa}$$

Is the behaviour of this alloy better described by the von Mises or the Tresca yield criterion? At what stress would the material yield in plane strain compression?

4. It is proposed to fabricate the propeller drive shaft for a human-powered aircraft from a thin-walled tube of an aluminium alloy, with a uniaxial yield stress of 400 MPa. The shaft is to be 20 mm in diameter. In service the shaft will be subjected to a driving torque of 40 Nm and a simultaneous axial compressive thrust from the propeller of 1600 N.

   Using the Tresca yield criterion, propose a minimum thickness of the tube wall, assuming a reasonable factor of safety and further assuming that the mode of failure will be general yielding.

5. Poly(methylmethacrylate) (PMMA) obeys a yield criterion of the modified von Mises type, in which the shear yield stress k depends on the hydrostatic stress, $\sigma_H = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$, in such a way that

$$k = 72 - 0.4\,\sigma_H \quad \text{MPa}$$

   Specify the forms of the yield surface for PMMA (i) in principal stress space and (ii) for plane stress.

   A tube of PMMA with mean radius 50 mm and wall thickness 2 mm is subjected to internal pressure. Show that the tube will yield when the internal pressure is increased to 4.1 MPa.

6. (a) A fully annealed isotropic sheet of Ti–6Al–4V is subjected to the following stress state:

$$\sigma_{ij} = \begin{pmatrix} -700 & 300 & 0 \\ 300 & 200 & 0 \\ 0 & 0 & -400 \end{pmatrix} \text{MPa}$$

   Use the von Mises yield criterion to determine if yielding will occur if the uniaxial yield stress of this material is 950 MPa. Would yielding be expected on the Tresca yield criterion?

   (b) For *anisotropic* materials, the von Mises yield criterion can be modified as follows:

$$F(\sigma_2 - \sigma_3)^2 + G(\sigma_3 - \sigma_1)^2 + H(\sigma_1 - \sigma_2)^2 = 1$$

   where the constants F, G and H define the degree of anisotropy. F, G and H are calculated from the uniaxial yield stresses X, Y and Z corresponding to the principal 1, 2 and 3 directions respectively using the formulae

$$G + H = \frac{1}{X^2} \qquad H + F = \frac{1}{Y^2} \qquad F + G = \frac{1}{Z^2}$$

   If $X = Z = 925$ MPa and $Y = 1020$ MPa for cold rolled Ti–6Al–4V sheet, with the principal stresses of the stress state in (a) defined so that $\sigma_1 > \sigma_2 > \sigma_3$, determine whether or not yielding occurs for the stress state described in (a).

7. Explain clearly why Lüders bands seen on sheet steel subjected to uniaxial tension form at an angle of 54.7° to the tensile axis.

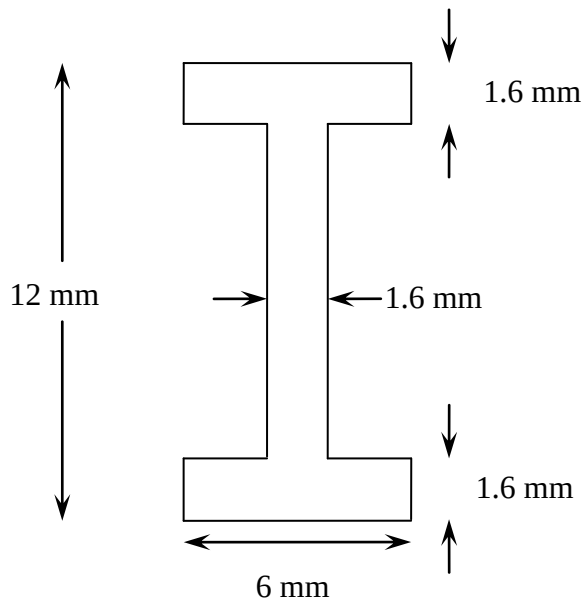
   A specimen of sheet steel is tested in unequal biaxial tension, and Lüders bands form at 60° to one of the tensile axes. What is the ratio between the two principal stresses in the plane of the sheet? If the greater of these two principal stresses is 500 MPa and the steel obeys the von Mises yield criterion, show that the yield stress in uniaxial tension is 458 MPa.

Question Sheet 2

1. What is meant by the efficiency of a deformation process? If the deformation is adiabatic (i.e., with no heat loss to the atmosphere or to the tooling), estimate the temperature rise that will result from extrusion of the materials in the table below, for an extrusion ratio of 3. Assume the efficiency of the extrusion process is 40% and that 95% of the work done in the process is converted to heat.

Material	Density ($Mg\ m^{-3}$)	Specific heat ($J\ kg^{-1}\ K^{-1}$)	Uniaxial yield stress (MPa)
Aluminium	2.7	900	100
Copper	9.0	385	150
Mild steel	7.9	450	300

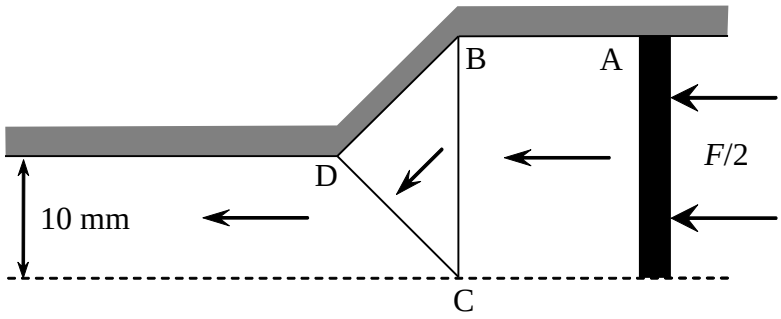
2. Use a work formula approach, show that the minimum pressure required to extrude aluminium curtain rail of I-section, 12 mm high with 6 mm wide flanges, all 1.6 mm thick, from 25 mm circular diameter bar stock, as shown in the diagram below, is 404 MPa.



Suggest a more realistic estimate using a suitable empirical formula for the extrusion pressure of extruded material.

[The mean uniaxial yield stress for aluminium for heavy deformation at room temperature is 150 MPa].

3. Half of a possible deformation pattern for the direct plane-strain extrusion of a metal slab is shown in the diagram below. The slab is initially 40 mm thick. It passes through a symmetrical 45° tapering die, with an extrusion ratio of 2. The angles BCD and CBD are both 45°. The distance AB is 15 mm. The width of the slab is 100 mm and its yield stress in pure shear is 150 MPa. Arrows indicate the direction of movement of material in each region.

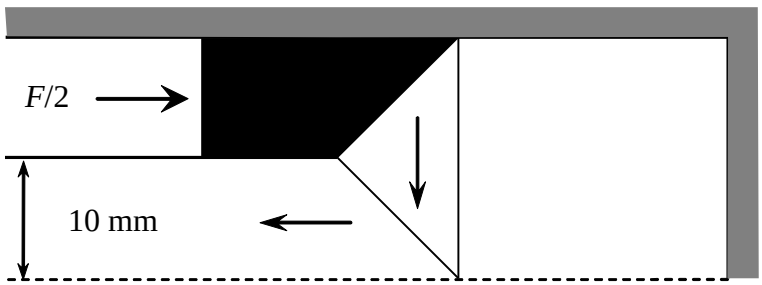


Show that upper bounds to the extrusion force F acting on the slab are

(i) 1.2 MN if the extrusion process is frictionless, and

(ii) 2.25 MN if sticking friction acts both on the die face BD and also on the wall of the die AB.

Compare your answers to (i) and (ii) with a simple work-formula estimate and comment on the reasons for any differences.



The same reduction in thickness could be achieved by indirect extrusion, as shown in the second diagram. Again, half the deformation pattern is shown and the arrows indicate the direction of movement of material. For this case, show that upper bounds to the extrusion force are

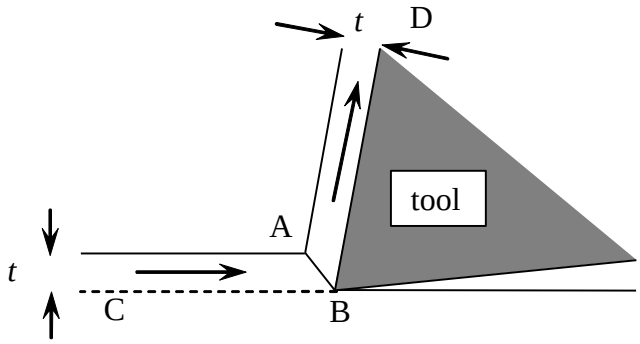
(iii) 1.2 MN if the extrusion process is frictionless, and

(iv) 1.8 MN if sticking friction acts on the die face and within the billet container.

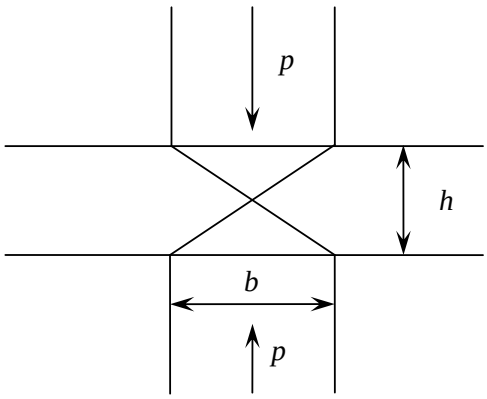
Hence comment on the possible advantages of indirect extrusion over direct extrusion.

4. In a simple model of an orthogonal machining operation a layer of thickness t is removed from a metal surface by a tool moving from right to left, as in the diagram below. The thickness of the chip formed is also t. Thus, relative to the tool, metal moves left to right, shearing on the plane BA. Friction can be ignored and the operation can be treated as one of plane strain. The angles ABC and ABD are both 50°.

   Draw a hodograph representing the relative motion of the layer and the chip. Calculate the force on the tool in its direction of motion, per unit width, for a metal with a shear yield stress of k. Hence show that the energy dissipated per unit volume of metal removed according to this model is 1.678k.



5. A possible deformation mode for the plane strain compression of a rigid-plastic plate, thickness h, between frictionless parallel punches for $h \approx b$ (i.e, h is similar to b, but not necessarily equal to b) is sohwn in the diagram below.



   Draw the corresponding hodograph and hence show that an upper bound to the pressure p on the punches needed to cause yielding in the plate is given by:

$$\frac{p}{2k} = \frac{1}{2}\left(\frac{h}{b} + \frac{b}{h}\right)$$

   where k is the yield stress of the plate in pure shear.

   A steel plate has a yield stress in the quenched and tempered condition which is 1.5 times its yield stress when normalised. Punches 10 mm wide are made from the quenched and tempered steel, and used to compress a plate of the same steel in the normalised condition. Show that the maximum thickness of plate which can be deformed plastically without the punches themselves yielding is 26.18 mm.

6. In the plane strain forging of square bar between smooth frictionless dies, slip occurs on planes at 45° to the forging direction, the planes forming a cross in the section of the bar. 90% of the energy expended in plastic deformation is dissipated as heat.

   By equating the rate of working of the forging force to the rate of heat generation in material passing through the shear planes, show that the localised temperature rise in steel during this forging operation is 107 K, assuming that the steel obeys the von Mises yield condition.

   (This is easier than it may at first appear. Sketch the deformation pattern and identify the planes on which shear occurs. Calculate the work done by the applied load when the bar is squashed by some small increment, δx, per platen. Then assume that 90% of this work is dissipated in heat in the volume of steel that has passed through the shear planes during this squashing operation.)

   [Uniaxial yield stress = 400 MPa; density = 7.8 Mg m^{-3}; specific heat = 0.5 kJ kg^{-1} K^{-1}]