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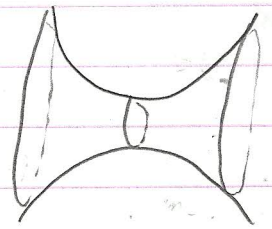
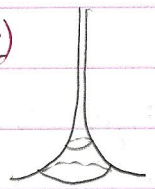
- Triângulos Ideais
- Fórmulas de Gauss Bonet
- Cilindros
  - ↳ 2 tipos de cilindros completos (I)
  - ↳ cilindro incompleto (II)

(I)



Uma única geodésica fechada que tem comprimento mínimo entre TODAS AS CURVAS FECHADAS (não homotopicamente triviais).

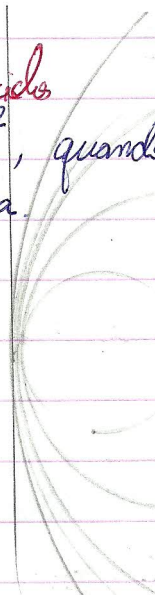
(II)



Não há geodésicas fechadas e há curvas fechadas não triviais de comprimento arbitrariamente pequenos.

Horociclo

Em $\mathbb{R}^2$, quando levamos o centro do círculo para ∞, o círculo → → reta.



E em $\mathbb{H}^2$ é análogo?

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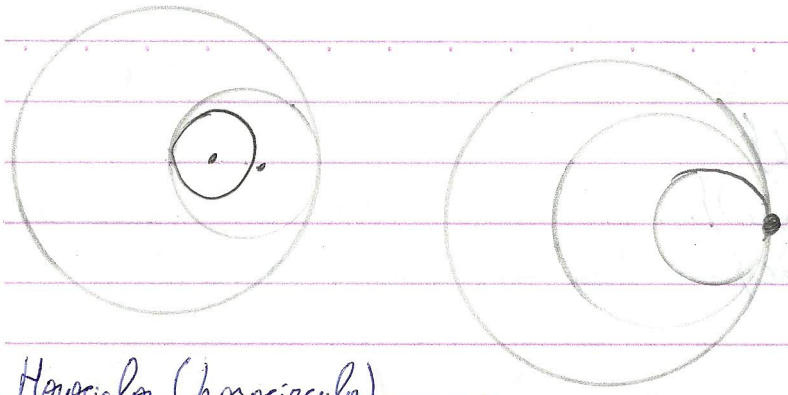
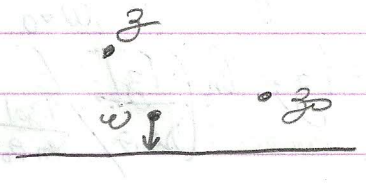


Horociclos (Horocírculos)

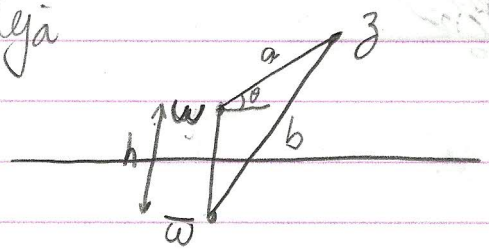
Horociclos (horocírculos)

Fato: $d_{\mathbb{H}^2}(z,\omega) = \ln\left(\dfrac{|z-\bar{\omega}|+|z-\omega|}{|z-\bar{\omega}|-|z-\omega|}\right)$

$\lim\limits_{\omega\to 0}\left[d_{\mathbb{H}^2}(z,\omega) - d_{\mathbb{H}^2}(z_0,\omega)\right] = (*)$ Graficamente:



Veja



Lei dos Cossenos:

$b^2 = a^2 + h^2 - 2ah\cos(\theta + \pi/2)$

$b^2 = a^2 + h^2 - 2ah\,\mathrm{sen}(\theta) \Rightarrow$

$b = (a^2 + h^2 - 2ah\,\mathrm{sen}\,\theta)^{1/2} =$

$= \left(a^2\left(1 + \frac{h^2}{a^2} - \frac{2h}{a}\mathrm{sen}\,\theta\right)\right)^{1/2} =$

$a\left(1 + (h/a)^2 - \frac{2h\,\mathrm{sen}\,\theta}{a}\right)^{1/2}$

Usando que $(1+x)^{\alpha} = 1 + \alpha x$ para x pequeno

$(h/a)^2 \to 0$ $\forall a$ pequeno

$b = a\left(1 + \frac{1}{2}\cdot\frac{2h\,\mathrm{sen}\,\theta}{a}\right) = a\left(1 + \frac{h\,\mathrm{sen}\,\theta}{a}\right) \Rightarrow b = a + h\,\mathrm{sen}\,\theta \Rightarrow$

$\boxed{b - a = h\,\mathrm{sen}\,\theta}$

Sendo assim, $(*) = \ln\left[\dfrac{|z-\bar{\omega}|+|z-\omega|}{|z-\bar{\omega}|-|z-\omega|}\cdot\dfrac{|z_0-\bar{\omega}|-|z_0-\omega|}{|z_0-\bar{\omega}|+|z_0-\omega|}\right] \underset{|\omega| \text{ peq}}{\simeq}$ indep/ω

$\ln\left[\dfrac{|z-\bar{\omega}|+|z-\omega|}{|z_0-\bar{\omega}|+|z_0-\omega|}\cdot\dfrac{\mathrm{sen}\,\theta_0}{\mathrm{sen}\,\theta}\right] \xrightarrow{\omega\to 0} \ln\left[\dfrac{|z|}{|z_0|}\,\dfrac{\mathrm{sen}\,\theta_0}{\mathrm{sen}\,\theta}\right] = \ln\left[\dfrac{|z|}{|z_0|}\,\dfrac{\mathrm{sen}\,\theta_0}{\mathrm{sen}\,\theta}\right]$

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Círculo em $\mathbb{H}^2$

$= \{z;\ d_{\mathbb{H}^2}(z,\omega) = const\}$

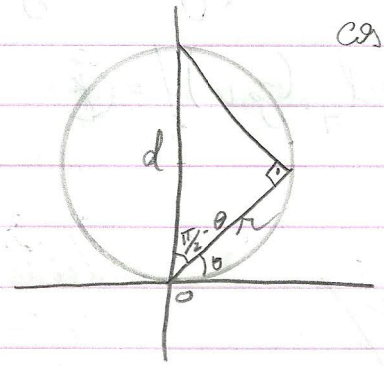
$= \{z;\ d_{\mathbb{H}^2}(z,\omega) = d_{\mathbb{H}^2}(z_0,\omega)\}$

$= \{z;\ \underbrace{d_{\mathbb{H}^2}(z,\omega) - d_{\mathbb{H}^2}(z_0,\omega)}_{\omega \to \xi \in \partial\mathbb{H}} = 0\}$

O conjunto $\{z;\ \lim_{\omega \to 0} d_{\mathbb{H}^2}(z-\omega) - d_{\mathbb{H}^2}(z_0,\omega) = 0\} =$

$= \{z;\ \ln\left(\frac{|z|}{\operatorname{sen}\theta} \Big/ \frac{|z_0|}{\operatorname{sen}\theta_0}\right) = 0$

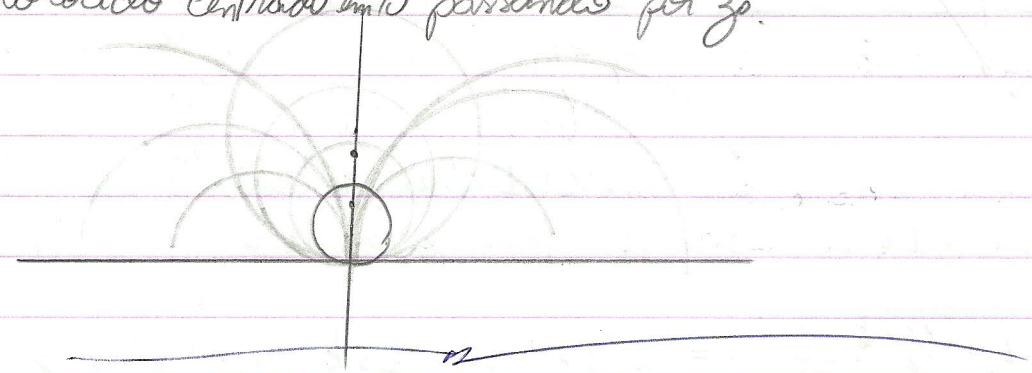
$\cos(\pi/2 - \theta) = \frac{r}{d}$



$\frac{|z|}{\operatorname{sen}\theta} = \frac{|z_0|}{\operatorname{sen}\theta_0} = cte = d$

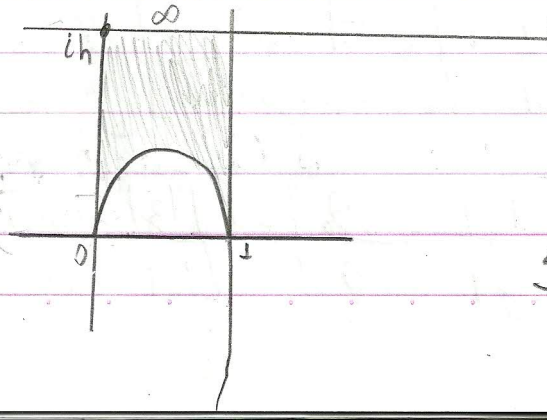
$\Downarrow$

horociclo centrado em 0 passando por z_0.



Triângulos ideais e horociclos

Horociclos centrados no ∞ = retas horizontais



C_h = horociclo centrado no ∞

Simetria de

$S(z) = \frac{1}{1-z}$

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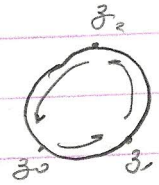
$S(0)=1,\ S(1)=\infty,\ S(\infty)=0 \Rightarrow S(z)=\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

Exercício $T: \mathbb{H} \to D$

$T \circ S \circ T^{-1}$ é uma rotação. Encontre T de ângulo $\frac{2\pi}{3}$



$$S \circ S(z) = S\left(\frac{1}{1-z}\right) = \frac{1}{1-\left(\frac{1}{1-z}\right)} = \frac{1}{\frac{\cancel{1}-z-\cancel{1}}{1-z}} = \frac{z-1}{z}$$

$$S \circ (S^2(z)) = S\left(\frac{z-1}{z}\right) = \frac{1}{1-\left(\frac{z-1}{z}\right)} = \frac{1}{\frac{\cancel{z}-\cancel{z}+1}{z}} = z$$

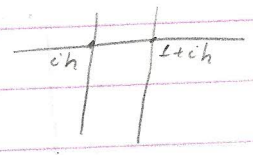
(Pto fixo 3 composição)

$$S(z)=z \Rightarrow \frac{1}{1-z}=z \Rightarrow z-z^2=1 \Rightarrow z^2-z+1=0$$

$$\Leftrightarrow \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Devemos ter $T\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) \to 0$

$S(C_\infty^h)$ = é um horociclo centrada em 0.



$$S(1+ih) = \frac{1}{\cancel{1}-\cancel{1}-ih} = \frac{-1}{ih} = i \cdot \frac{1}{h} = C_0^{1/h}$$

$$S(C_0^{1/h}) = S^{-1}(C_\infty^h) = C_h^{1/h}$$

$$\omega = \frac{1}{1-z} \Leftrightarrow 1-z = \frac{1}{\omega} \Leftrightarrow z = 1-\frac{1}{\omega}, \quad S^{-1}(\omega) = 1-\frac{1}{\omega} = \frac{\omega-1}{\omega}$$

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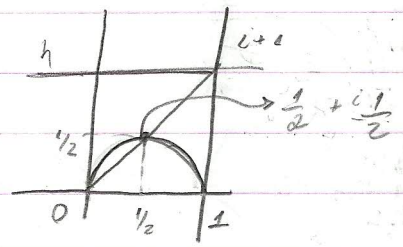
$S^{-1}(C_\infty^h)$ = horociclo centrado em ∞

$$S^{-1}(ih) = \frac{ih-1}{ih} = 1 - \frac{1}{ih} = 1+ih$$

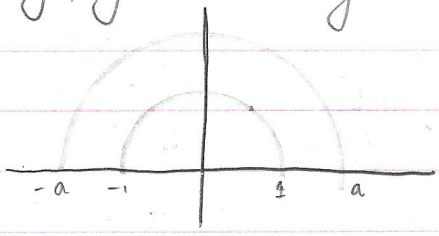
Se $h=1$, C_0, C_1, C_∞ são tangentes duas a duas.

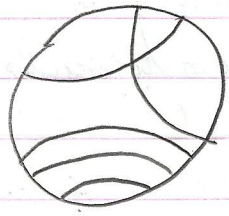


$$S\left(\frac{1}{2}+\frac{i}{2}\right) = \frac{1}{1-\left(\frac{1}{2}+\frac{i}{2}\right)} = \frac{1}{\frac{2-(1+i)}{2}} = \frac{2}{1-i} = \frac{2(1+i)}{2} = 1+i$$

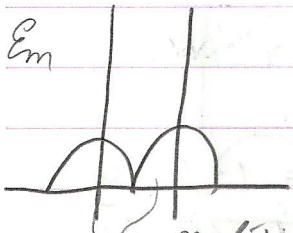
Lema Duas geodésicas $\neq$, em $\mathbb{H}$ tem uma perpendicular comum $\Leftrightarrow$ elas são disjuntas em $\mathbb{H}\cup\hat{\mathbb{R}}$, isto é, se e só se são disjuntas em $\mathbb{H}$ e são extremos não assintóticos. Neste caso a ≥ 1 é única.

Prova Se g_1, g_2 são disjuntas em $\mathbb{H}\cup\hat{\mathbb{R}} \Rightarrow$ por uma isometria





Em



geodésica centrada nos eixos verticais, por isso não da certo.

As geodésicas são separadas.

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$$\frac{w-\alpha}{w-\beta} = \frac{\lambda(z-\alpha)}{(z-\beta)} \Rightarrow w(z) = \frac{(\ )z+(\ )}{(\ )z+(\ )}$$

$w(\alpha)=\alpha$ $\quad T(\xi) = \frac{(\xi-\alpha)}{(\xi-\beta)}$

$w(\beta)=\beta$ $\quad w(z) = T^{-1} \circ \psi \circ T(z)$

$w'(\alpha)=\lambda$ $\quad T(w) = \lambda T(z)$

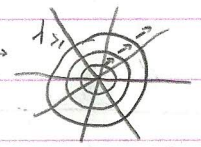
$w'(\beta) = 1/\lambda$ $\quad \psi(z) = \lambda z$

$\psi\left(\frac{az+b}{cz+d}\right)$

$T^{-1}\left(\frac{\lambda(az+b)}{cz+d}\right)$

$\psi(z) = \lambda z$ → hiperbólica

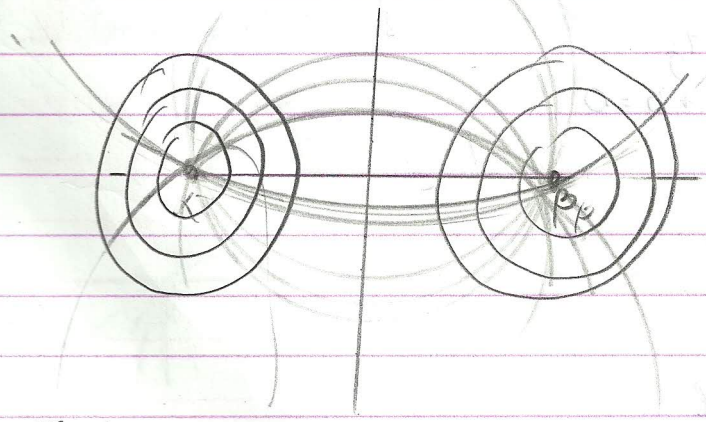
$\lambda \in \mathbb{R}$ → as coisas são expandidas ou reduzidas →



· $\lambda = 1$ (elíptica

$\lambda \in \mathbb{C}$ em geral → $\lambda = e^{i\theta}$ → os círculos são preservados

Fazendo a transformação $\alpha \to 0$, $\beta \to \infty$ temos retas que passam por 0 e ∞, então é a família de círculos que passam por α e β. Além disso temos os círculos onde os "centrados" em "a" são os inversos dos "centrados" em "b".



Círculos de Apolônio

$T(z) = \frac{az+b}{cz+d}$ $\quad a,b,c,d \in \mathbb{C}$ $\quad$ Pto fixo: $T(z) = z \Rightarrow$

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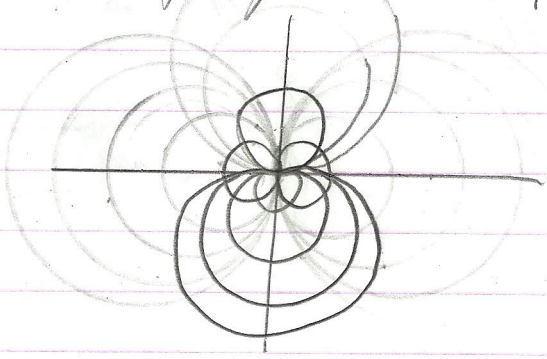
$$\frac{az+b}{cz+d} = z \Rightarrow az+b = cz^2+dz \Rightarrow cz^2+(d-a)z-b=0$$

2 raizes $\Rightarrow$ dois ptos fixos (fig anterior, α, β são fixos
1 raiz $\Rightarrow$ 1 pto fixo (α e β vão colidir)



$T(z) = z+b$

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Pontos fixos de uma TLF:

$$T(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc=1$$

$$\frac{az+b}{cz+d} = z \iff az+b = cz^2+dz$$
$$cz^2+(d-a)z \quad b=0$$

$$z = \frac{a-d}{2c} \pm \frac{1}{2c}\sqrt{(d-a)^2+4cb}$$

Note que:

$$d^2 - 2ad + a^2 + 4bc + 4ad - 4ad$$
$$d^2 + 2ad + a^2 + 4bc - 4ad$$
$$(d+a)^2 - 4(\underbrace{ad-bc}_{=1}) = (d+a)^2 - 4$$