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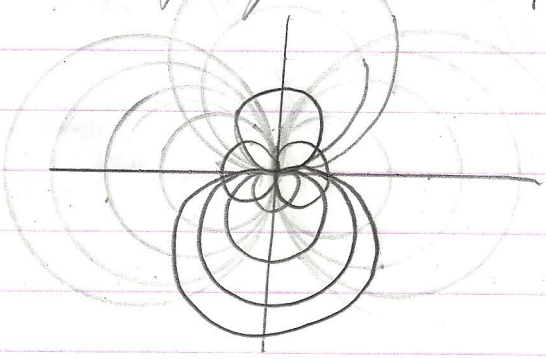
$$\frac{az+b}{cz+d} = z \Rightarrow az+b = cz^2+dz \Rightarrow cz^2+(d-a)z-b=0$$

2 raízes ⇒ dois pts fixos (fig anterior, α, β são fixos
1 raiz ⇒ 1 pto fixo (α e β vão colidir)



$T(z) = z+b$

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Pontos fixos de uma TLF:

$$T(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc=1$$

$$\frac{az+b}{cz+d} = z \iff az+b = cz^2+dz$$
$$cz^2+(d-a)z-b=0$$

$$z = \frac{a-d}{2c} \pm \frac{1}{2c}\sqrt{(d-a)^2+4cb}$$

Note que:

$$d^2 - 2ad + a^2 + 4bc + 4ad - 4ad$$
$$d^2 + 2ad + a^2 + 4bc - 4ad$$
$$(d+a)^2 - 4(ad-bc) = (d+a)^2 - 4$$
$$ad-bc=1$$

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Então

$$z = \frac{a-d}{2c} \pm \frac{1}{2c}\sqrt{(a+d)^2-4}$$

Suponha que $a, b, c, d \in \mathbb{R} \Rightarrow T(z)$ tem pontos fixos:

i) 2 pontos fixos distintos e reais $\Leftrightarrow (a+d)^2 > 4$ ← T é chamada Hiperbólica

ii) 1 ponto fixo apenas $\Leftrightarrow (a+d)^2 = 4$ ← T é chamada Parabólica

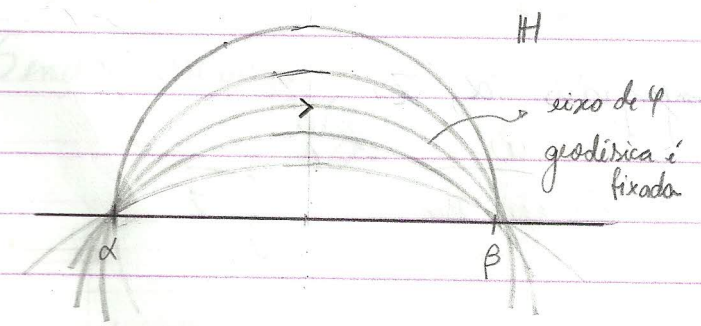
iii) 2 pontos fixos complexos conjugados $\Leftrightarrow (a+d)^2 < 4$ ← T é chamada Elíptica

Se φ é TLF (qualquer) tem dois pontos fixos distintos então φ pode ser escrita $\varphi(z) = w$ onde

$$\frac{w-\alpha}{w-\beta} = \lambda \frac{z-\alpha}{z-\beta}$$

Se $\varphi \in PSL(2,\mathbb{R})$

i) Pontos fixos reais $\alpha, \beta \in \mathbb{R}$



Se $\lambda > 1$

$$\lambda = \varphi'(\alpha) = \frac{1}{\varphi'(\beta)}$$

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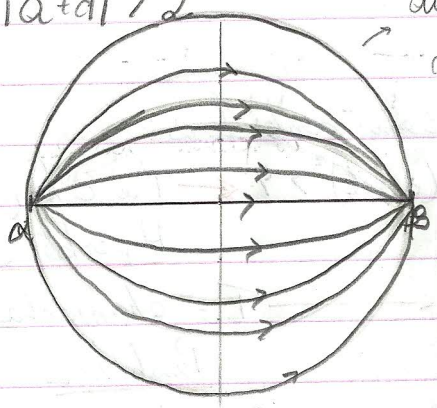
No disco unitário:

$(a+d)^2 > 4$

$|a+d| > 2$



distâncias constante da geodésica central

T hiperbólica

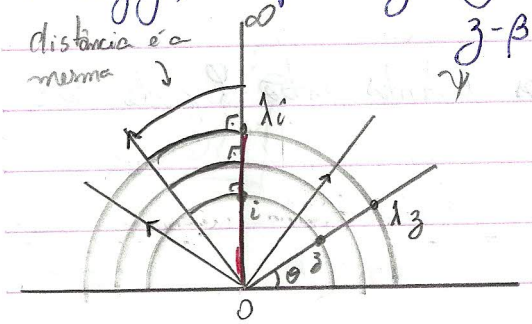
T é a análoga hiperbólica de uma translação.

$T^n(z) \to \beta, \ n \to \infty$

$\to \alpha, \ n \to -\infty$

Por conjugação (por $h(z) = \frac{z-\alpha}{z-\beta}$)

$h \circ \varphi \circ h^{-1}$



$\psi(z) = h^{\mp 1} \circ \varphi \circ h^{\mp 1}(z) = \lambda z$

$\inf \{ d(z, \lambda z) \} = d(iy, \lambda iy)$

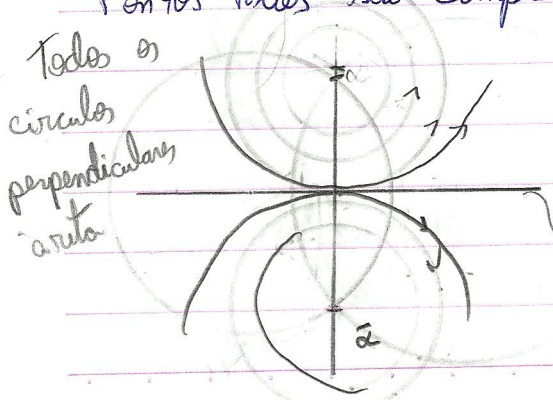
isto é, este ínfimo acontece exatamente para pontos na geodésica de 0 a ∞, chamada eixo de ψ

ψ não tem ponto fixo.

iii) $(a+d)^2 < 4$ elíptico

Pontos fixos são complexos conjugados $\alpha, \bar{\alpha}$

$$\frac{w-\alpha}{w-\bar{\alpha}} = \lambda \frac{z-\alpha}{z-\bar{\alpha}}$$



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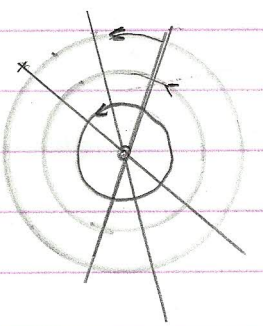
por $h(z) = \frac{z-\alpha}{z-\bar{\alpha}}$ manda $\alpha \to 0$, $\bar{\alpha} \to \infty$ Vira exatamente rotações $\psi(z) = h^{\pm 1} \circ \varphi \circ h^{\mp 1} = e^{i\theta} z$.



Como as famílias são preservadas as retas são levadas umas para outras

$\psi(z)$ → 1 ponto fixo.

ii) Caso Parabólico $(a+d)^2 = 4$ ou $|a+d| = 2$

Apenas 1 ponto fixo

$h(z) = \frac{1}{z-\alpha} \longrightarrow \psi(z) = h \circ \varphi \circ h^{-1}(z)$

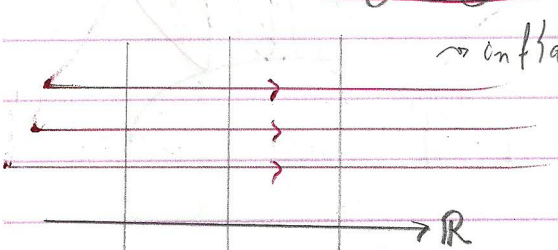
$\psi(z)$ é uma TLF sem pontos fixos em $\mathbb{C}$ (tinha apenas um (α) e mandamos $\alpha \to \infty$).

$\psi(z) = z + b$, $b \in \mathbb{R}$ → Note que devemos ter $c = 0$ pois caso contrário cairíamos numa equação do 2° grau, e $a = 1$ pois caso contrário $az + b = z \Rightarrow z = \frac{b}{a-1}$ é pto fixo.

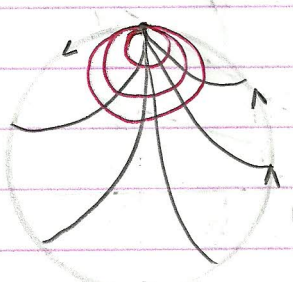
Sendo assim:

$\boxed{\psi(z) = z + b}$

→ $\inf\{|dz, \varphi(z)|\} \neq 0$



conforme vai para cima, z se aproxima de $\psi(z)$



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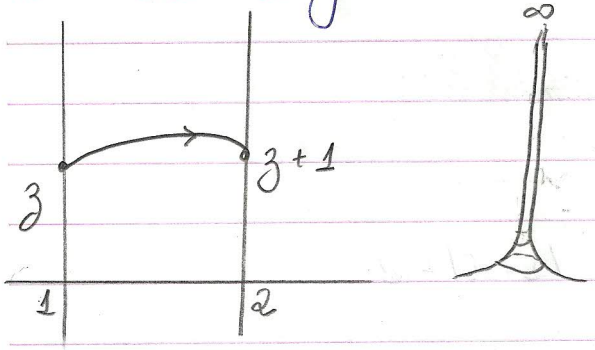
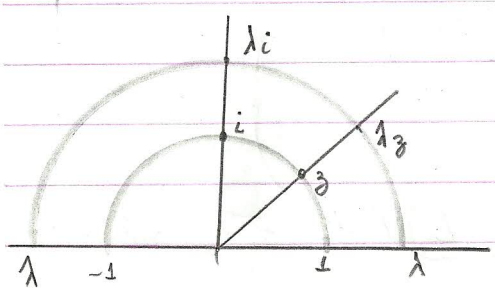
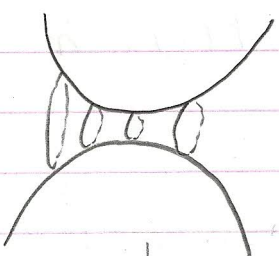
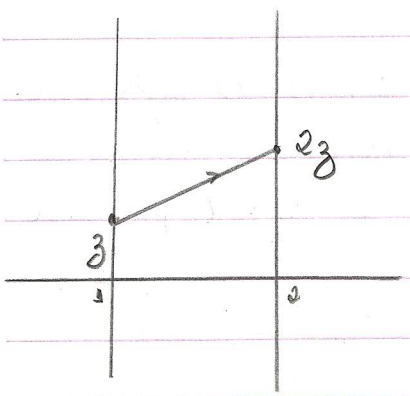
Então, analisando $|a+d|$ decidimos em qual caso estamos, contanto que $ad-bc=1$.

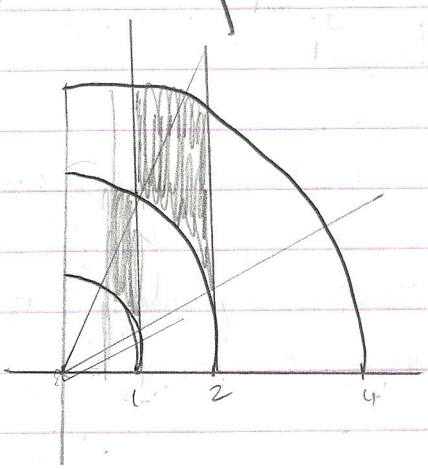
Voltando às colagens.

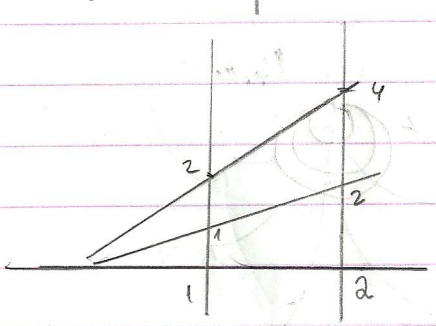


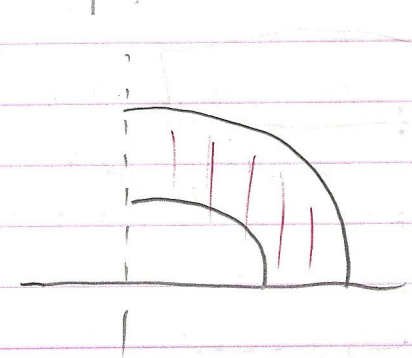


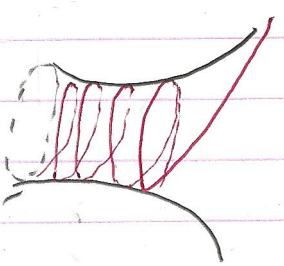












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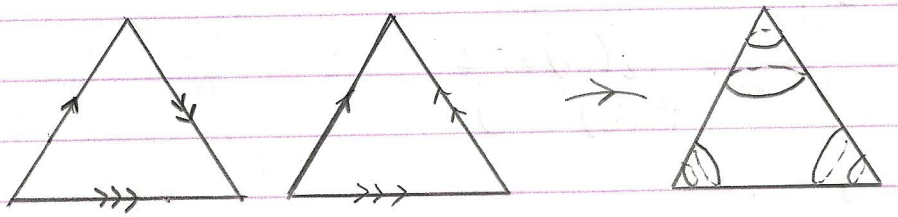
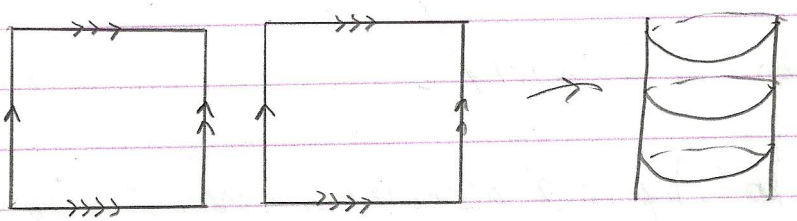
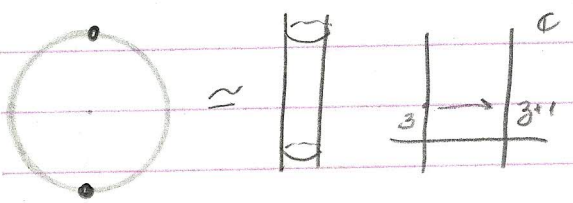
A esfera menos 3 pontos

não é possível fazer no plano euclidiano

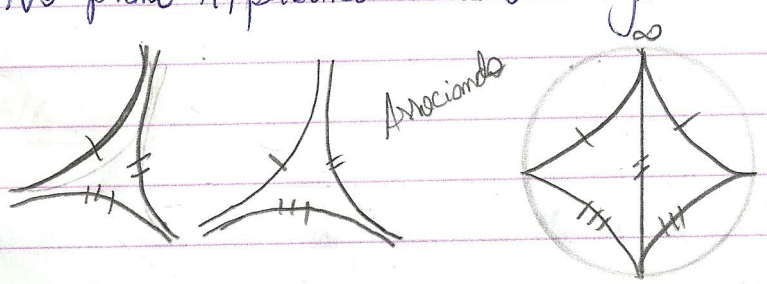


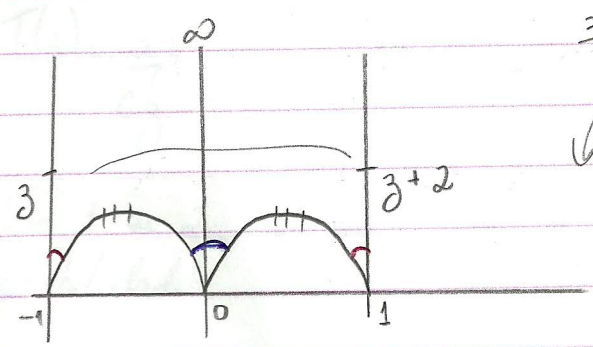


A esfera menos 2 pontos



No plano Hiperbólico temos o triangulo ideal





3 geodésicas saindo do infinito

Quero colar -1 em 1

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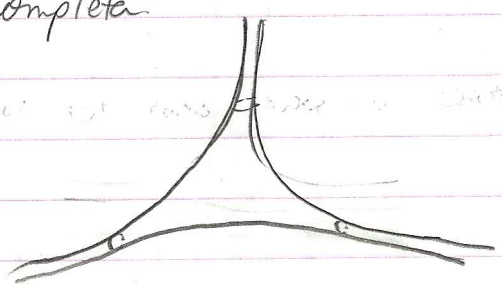
$\varphi_1(z) = z + 2 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad ad - bc = 1$

$\varphi_2(-1) = 1$

$\varphi_2(z) = \dfrac{z}{2z+1} \rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad \varphi_2(0) = 0$

Nos dois casos são parabólicas $\Rightarrow a + d = 2$.

É verdade (provar mais tarde) que o quociente dessas identificações é uma esfera hiperbólica menos 3 pontas (estão no ∞) completa

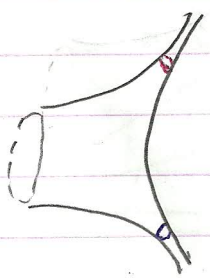


$\varphi(z) = az + b \rightarrow$ queremos $\varphi(-1) = 1$ então

$-a + b = 1 \Rightarrow b = 1 + a$

$\Rightarrow \varphi(z) = az + (1+a)$

Se $a \neq 1$, o quociente é algo como



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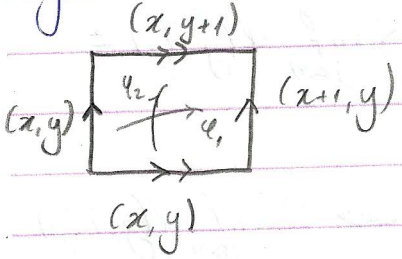
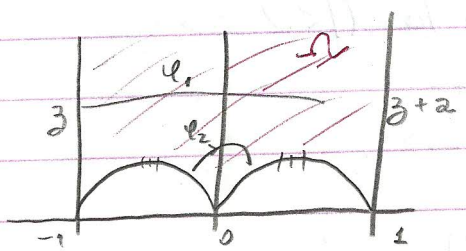
Veja no caso do toro



$\varphi_1(x,y) = (x+1, y)$
$\varphi_2(x,y) = (x, y+1)$

geram um grupo isomorfo a $\mathbb{Z}^2$

Voltando...



Considere o subgrupo de $Isom^+(\mathbb{H})$ gerado por φ_1, φ_2 e considere todos os transladados de Ω por elementos de T. (É análogo ao que acontece no caso do toro

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18/10/2011 } Seminário
24/10/2011

25/10/2011

Pontos Fixos da transformação de Möbius em $\hat{\mathbb{C}}$

$$T(z) = \frac{az+b}{cz+d} \quad ; \quad T(z) = z$$

tem 1 ou 2 soluções. Se $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ há 3 possibilidades: