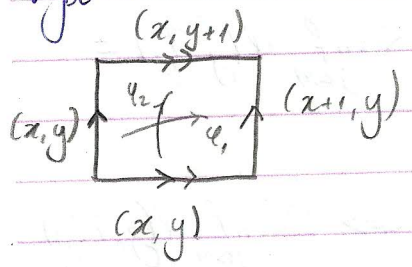
17/10/11

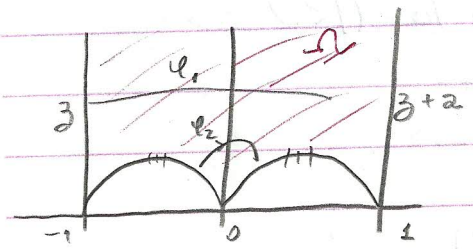
Veja no caso do toro



$\varphi_1(x,y) = (x+1, y)$
$\varphi_2(x,y) = (x, y+1)$

geram um grupo isomorfo a $\mathbb{Z}^2$

Voltando...



Considere o subgrupo de $Isom^+(\mathbb{H})$ gerado por φ_1 e φ_2 e considere todos os translados de Ω por elementos de T. (É análogo ao que acontece no caso do toro

18/10/2011 } Seminário
24/10/2011

25/10/2011

Pontos Fixos da transformação de Möbius em $\hat{\mathbb{C}}$

$$T(z) = \frac{az+b}{cz+d} \quad : T(z) = z$$

tem 1 ou 2 soluções. Se $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ há 3 possibilidades:

25 / 10 / 2011

i) Hiperbólica: 2 ptos fixos reais distintos $\Leftrightarrow 0 < \inf_{z \in H} d(z, Tz) = \min_{z \in H} d(z, Tz) \Leftrightarrow |a+d| > 2$

ii) Parabólica: 1 ponto fixo real $\Leftrightarrow |a+d| = 2 \Leftrightarrow 0 = \inf_{z \in H} d(z, T(z))$ e não é realizada.

iii) Elítica: 2 pontos fixos complexos conjugados $\Leftrightarrow |a+d| < 2 \Leftrightarrow$
$\Leftrightarrow 0 = \inf_{z \in H} d(z, T(z)) = \min_{z \in H} d(z, T(z))$.

i) Por uma troca de coordenadas em $Möb(\hat{\mathbb{C}})$ podemos assumir que os pontos fixos são 0 e $\infty \Rightarrow T(z) = \lambda z$, $\lambda > 0$.

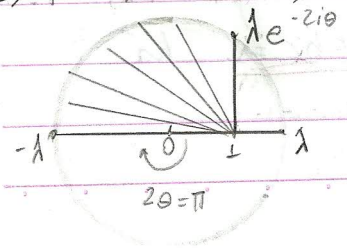
$$d(z, \lambda z) = \ln \frac{|z - \lambda \bar{z}| + |z - \lambda z|}{|z - \lambda \bar{z}| - |z - \lambda z|} = \ln \frac{|1 + \lambda \frac{\bar{z}}{z}| + |1 - \lambda|}{|1 - \lambda \bar{z}/z| - |1 - \lambda|} =$$

$$= \ln \frac{|1 + \lambda e^{-2i\theta}| + |1 - \lambda|}{|1 - \lambda e^{-2i\theta}| - |1 - \lambda|}$$

Então $\inf_{z \in H} d(z, \lambda z) = \inf_{\substack{0 < \theta < \pi \\ \text{semi-plano superior}}} \ln \frac{|1 - \lambda e^{-2i\theta}| + |1 - \lambda|}{|1 - \lambda e^{-2i\theta}| - |1 - \lambda|}$

Note que se $f(x) = \frac{x+c}{x-c} \Rightarrow f'(x) = \frac{(x-c)-(x+c)}{(x-c)^2} = \frac{-2c}{(x-c)^2}$, se $c > 0$

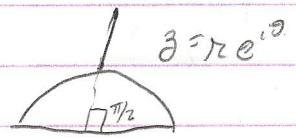
$\Rightarrow f'(x) < 0 \Rightarrow$ o mínimo de f acontece no máximo de x.



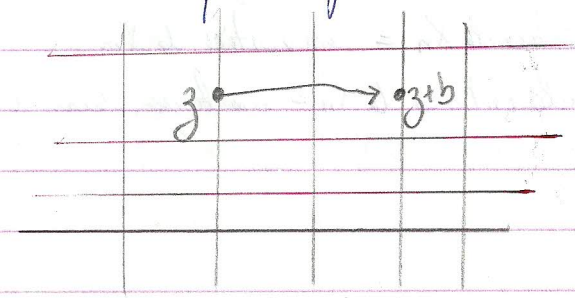
$$\max |1 - \lambda e^{-2i\theta}| = |1 - \lambda e^{-i\pi}| \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

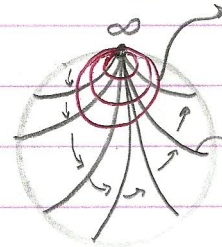
$$\boxed{e^{-i\pi} = -1}$$

$$\Rightarrow \inf_{z\in H} d(z, \lambda z) = \ln \frac{|1+\lambda| + |1-\lambda|}{1+\lambda - |1-\lambda|} = d(iy, T(iy)) \quad \forall y \in \mathbb{R}$$



ii) Por uma troca de coordenadas em Möb $(\hat{\mathbb{C}})$ podemos assumir que o ponto fixo é $\infty \Rightarrow T(z) = z + b$



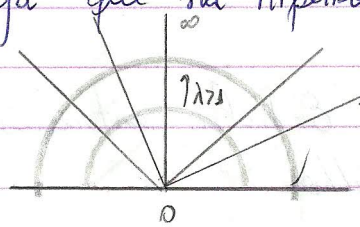


horociclos são preservados

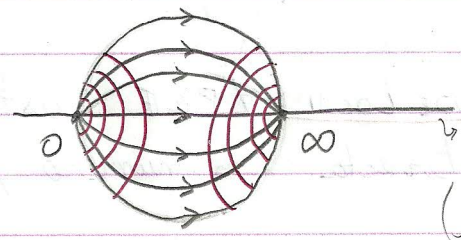
no plano, são as paralelas ////

$$d(z, (z+b)) \leq \frac{|b|}{y} \Rightarrow \text{quanto maior o } y, \text{ a dist. tende a } 0.$$

Veja que na hiperbólica, temos:



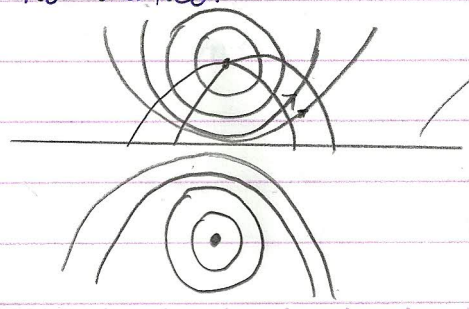
são preservados



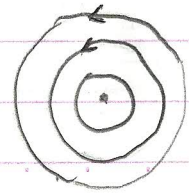
eixo de translação (a perpendicular em comum)

Saem do 0 e vão para ∞

E no elítico.



preserva o eixo real $\Rightarrow$ rotação hiperbólica. Nenhuma reta hiperbólica é preservada



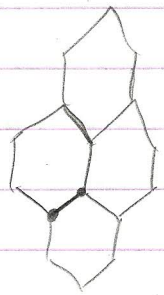
25/10/11

Tecelagens ou ladrilhamento

$\mathbb{E} = \mathbb{R}^2, \mathbb{H}^2, S^2$

Def. Um ladrilhamento de $\mathbb{E}$ é uma coleção de polígonos $X_n \subseteq \mathbb{E}$, chamados ladrilhos, satisfazendo:

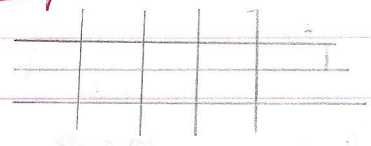
(1) $\bigcup X_n = \mathbb{E}$ e $X_n \cap X_m = \begin{cases} \emptyset \\ \text{Um vértice de } X_n = \text{um vértice de } X_m \\ \text{Um lado inteiro de } X_n = \text{um lado inteiro de } X_m \end{cases}$



(2) Os X_n são todos isométricos a uma coleção finita de ladrilhos modelos (Tipicamente 1 só).

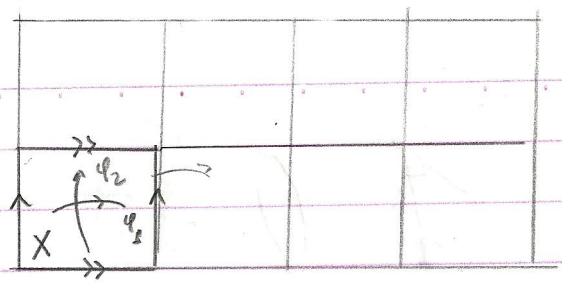
(3) <u>FINITUDE LOCAL</u>: $\forall P \in \mathbb{E}$ existe $\varepsilon > 0$ tq a ε-bola centrada em P intersecta um número finito de ladrilhos

Exemplos

0) 

Note que no toro temos:

25/10/11

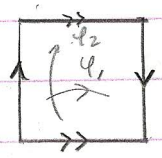


φ_1 e φ_2 são isometrias que realizam as colagens no bordo de X

grupo Γ [Isom $(\mathbb{R}^2) = \Gamma$ = grupo gerado por φ_1 e φ_2.

↳ grupo de ladrilhamento associado a X e colagens

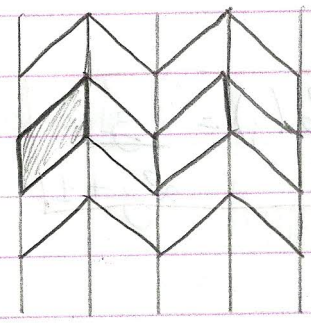
1) Garrafa de Klein



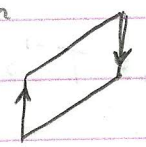
→ Mesmo ladrilhamento anterior

$\varphi_1(x,y) = (x+1, 1-y)$
$\varphi_2(x,y) = (x, y+1)$

1')

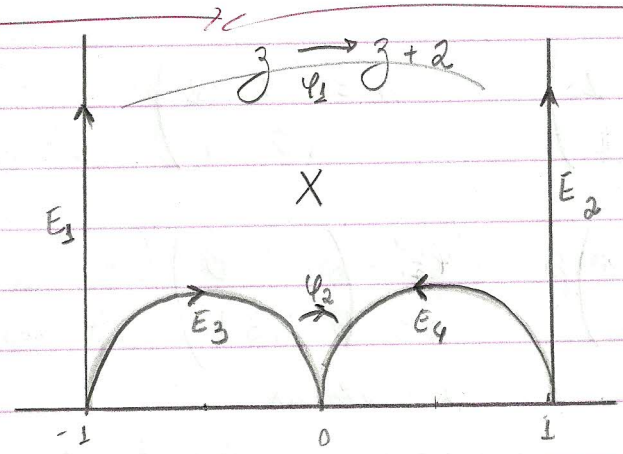


Reflexão, começamos com

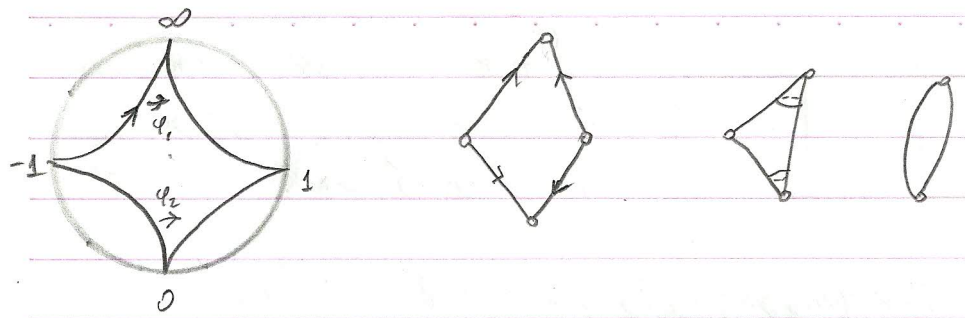


Tecelagens de $\mathbb{H}$

$S^2 \setminus 3$ pontos



25 / 10 / 11



Fórmulas

$\varphi_1(z) = z + 2$ ← Parabólica

$T^{-1}(z) = T(z) = \frac{-1}{z}$ T leva: $0 \to \infty$
$\infty \to 0$
$-1 \leftrightarrow 1$

Sendo assim
$T(E_3) = E_2$
$T(E_4) = E_1$

Defino $\varphi_2 = T^{-1} \circ \varphi_1^{-1} \circ T(z) = \frac{-1}{-1/z - 2} \Rightarrow \boxed{\varphi_2(z) = \frac{z}{2z+1}}$

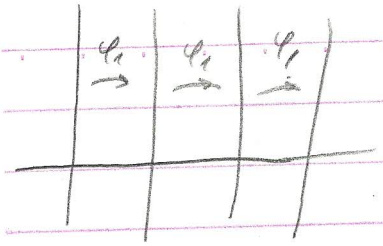
As matrizes associadas

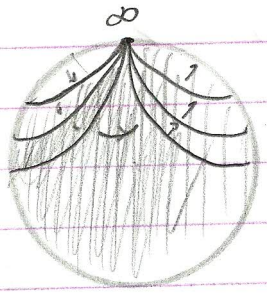
Transposta

$\varphi_1 \to \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\varphi_1^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ $\varphi_1^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$

$\varphi_2 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ $\varphi_2^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ $\varphi_2^m = \begin{pmatrix} 1 & 0 \\ 2m & 1 \end{pmatrix}$

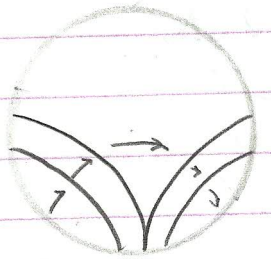
as inversas
continuam sendo inteiras.

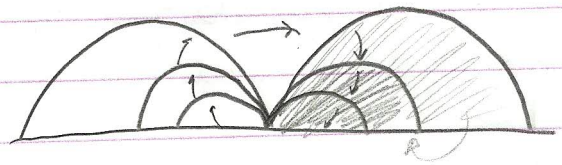




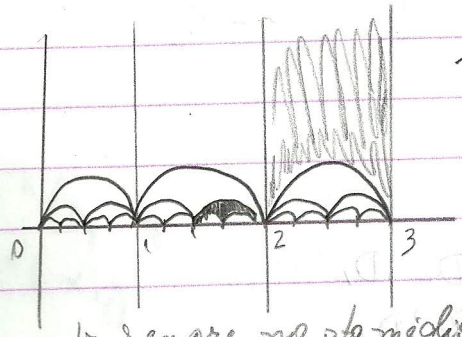
note que toda parte é levada na seguinte.

O mesmo ocorre com a parte debaixo





Ladrilhamento diádico de $\mathbb{H}^2$ por triângulos ideais

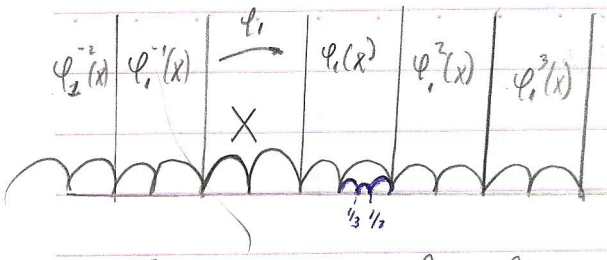


Todos triângulos ideais → 3 ptos no infinito

↳ sempre no pto médio

Obs.: Qualquer transformação da forma $\varphi(z) = az + (a+1)$ é uma isometria que identifica E_1 com E_2, $\forall a > 0$

25 / 10 / 11



$\varphi_2(\infty) = 1/2$
$\varphi_2(1) = 1/3$

não é diádico

Γ = grupo de ladrilhamento $[PSL(2,\mathbb{Z}) = \langle \varphi_1, \varphi_2 \rangle$

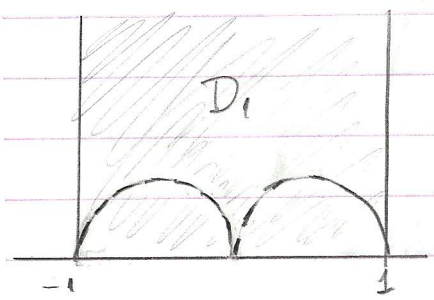
subgrupo

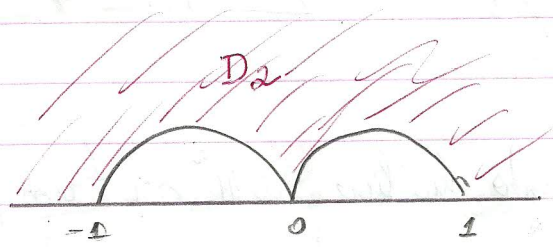
Devemos mostrar que:

1. $\Gamma(X) = \bigcup_{\varphi \in \Gamma} \varphi(X)$ é um ladrilhamento de $\mathbb{H}^2$

Γ é um grupo livre em 2 geradores

↳ chega-se na Id de modo trivial, qualquer outro produto não chega na Id.





Obs 1) $D_1 \cap D_2 = X$ ($\pm$ bordo)

Obs 2) $D_1 \cup D_2 = \mathbb{H}$

Obs 3) $\varphi_1^n(D_1) \cap D_1 = \emptyset \quad \forall n \neq 0 \Rightarrow \varphi_1^n(D_1) \subseteq D_2 \setminus D_1$

Obs 4) $\varphi_2^m(D_2) \cap D_2 = \emptyset \quad \forall m \neq 0 \Rightarrow \varphi_2^m(D_2) \subseteq D_1 \setminus D_2$

para $n_1 \neq 0$

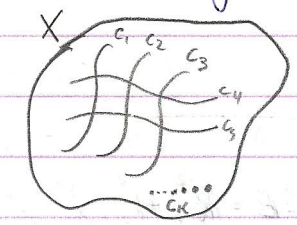
$$\left.\begin{array}{ll} \varphi_1^{n_1}(X) \subseteq \varphi_1^{n_1}(D_1) \subseteq D_2 \setminus D_1 & (\cap X = \emptyset) \\ \varphi_2^{m_1}(\varphi_1^{n_1}(X)) \subseteq \varphi_2^{m_1}(D_2) \subseteq D_1 \setminus D_2 & (\cap X = \emptyset) \\ \varphi_1^{n_2}(\varphi_2^{m_1}(\varphi_1^{n_1}(X))) \subseteq \varphi_1^{n_2}(D_1) \subseteq D_2 \setminus D_1 & \end{array}\right\} \Rightarrow$$

25/10/11

$$\Rightarrow \underbrace{\varphi_2^{m_k}\varphi_1^{n_k}\varphi_2^{m_{k-1}} \cdots \varphi_2^{m_1}\varphi_1^{n_1}(X)}_{\neq Id} \cap X = \emptyset$$

01/11/11

- (X, d) espaço métrico
- R relação simétrica e reflexiva em X ($\Leftrightarrow$ R = coleção de subconjuntos (QUAISQUER) de X cuja união é X e $xRy \Leftrightarrow x, y \in$ mesmo conjunto de R)

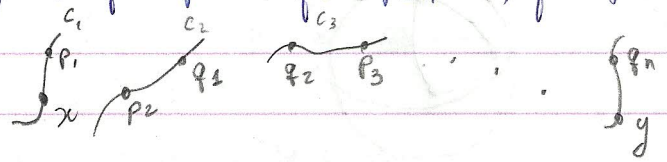


$$xRy \Leftrightarrow \begin{cases} x, y \in C_i \\ x = y \end{cases}$$

(Lembre que todo espaço métrico é Hausdorff)

Definimos R-cadeias de x a y é a sequência $\{(p_i, q_i)\}$

$$xRp_1,\ q_1Rp_2,\ q_2Rp_3,\ \ldots,\ q_iRp_{i+1},\ \ldots,\ q_nRy$$



$$L^R((p_i, q_i)) = \Sigma\, d(p_i, q_i)$$
$$d^R(x, y) = \inf\{L^R(p_i, q_i) : (p_i, q_i) \text{ é } R\text{-cadeia de } x \text{ a } y\}$$

<u>Lema</u>: d^R é uma semi-métrica, isto é, satisfaz todas as propriedades de uma métrica exceto possivelmente

$$d^R(x, y) = 0 \underset{\text{necessariamente}}{\not\Rightarrow} x = y$$